A SIMPLE APPROACH TO THE STUDY OF WAVE PATTERNS

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SUMMARY

The paper revisits some pioneering work of Sir Thomas Havelock on wave patterns with particular attention focussed on his graphical method of analysis. Motivated by a desire to explore this method further using numerical methods, it is extended in a simple manner to give three-dimensional illustrations of the wave patterns of a point disturbance in deep and shallow water. All results are confined to the sub- and trans-critical regimes with some obtained very close to the critical Depth Froude Number. Some conclusions are drawn on the wave types produced when operating close to the critical speed and their decay with distance off.

NOMENCLATURE

a	wave amplitude
c	wave celerity
g	gravitational acceleration
h	water depth
k	wave number
n	wave decay index
T	wave period
β	wave train azimuth angle
γ	wave decay constant
ζ	local wave amplitude
λ	wavelength
ξ	wave decay distance
ω	angular wave frequency

1. INTRODUCTION

The motivation for the work described in this paper was born out of curiosity stemming from the papers of a pioneer in the study of ship wave theory, Sir Thomas Havelock.

In 1934, he placed a paper before this Institution [1] aimed at helping practicing naval architects understand the mathematics of ship wave patterns and wave resistance. Early in the paper, Havelock introduced a simple graphical method, based on Kelvin's work with superposition of plane wave trains, which he showed could be used to generate the well-known Kelvin wave pattern. However, Havelock did not show the full development diagram for the method because "…there is too much detail for reproduction on a small scale, but it is interesting to see the picture of a familiar wave pattern emerging [1]"

It was Havelock's missing development diagram that provided the starting point for this paper. In addition, this simple graphical method lends itself to exploration on a computer, and could perhaps be used as an introduction to the fundamentals of vessel wave patterns in both deep and shallow water. Of particular interest is the behaviour of wave systems near the critical speed, itself of interest in the investigation of wash nuisance.

2. DISPERSIVE WAVES

It is assumed throughout this paper that the waves concerned are dispersive. This means that the velocity of a wave front depends on the wave frequency/period/length. The key equation linking wave speed and frequency is called the *linear dispersive relationship* and is given by:

$$\omega^2 = 2\pi g.\tanh(2\pi h/\lambda)/\lambda \qquad (1)$$

in water of depth h, which reduces to

$$\omega^2 = gk \qquad (2)$$

in deep water. Angular wave frequency is given by ω, wavelength by λ and "wave number" by k which is related to wave length by:

$$k = 2\pi/\lambda \qquad (3)$$

These equations are fundamental to the understanding of ship wave patterns. For example, the tanh() function in the dispersive relationship tends to unity as water depth h increases to infinity, resulting in the well-known relation for the deep water wave celerity, c:

$$c = (g\lambda/2\pi)^{1/2} \qquad (4)$$

because, in this case

$$c = \lambda/T = \omega/k \qquad (5)$$

Equations (1) to (5) will be very familiar to naval architects dealing with matters related to wave patterns (or "wave wash") and the nuisance these may cause in certain circumstances. These equations lie at the heart of what now follows.

3. WAVE PATTERNS

When a ship moves at a steady speed in calm deep water, the normal pressures over the hull are greatest over the upper part of the underwater body and result in a distinctive and well-known pattern of waves on the water surface [2].

At the stem of a conventional hull with no wave-cancelling device such as a bulb, the bow wave system starts with a crest and develops into two systems: the transverse and diverging systems. The transverse system has crests and troughs roughly normal to the direction of motion of the hull which move astern along the hull. Once a steady state has been reached, the wavelength of this system has been shown, using model measurements [2], to agree well with equation (4) for wave speed. As this is equal to ship speed, the dispersive nature of the transverse wave system is confirmed experimentally.

In concert with the transverse wave system is the diverging wave system which radiates out from the bow. This produces a series of quite short wave crests which follow the dispersive relationship and move along a mean line at an angle to the sailing line. The lines of transverse wave crests bend round as they spread out until they match direction, and coalesce to form *cusps,* with the diverging waves at the *cusp line.* Figure 1, from [2], shows this in diagrammatic form:

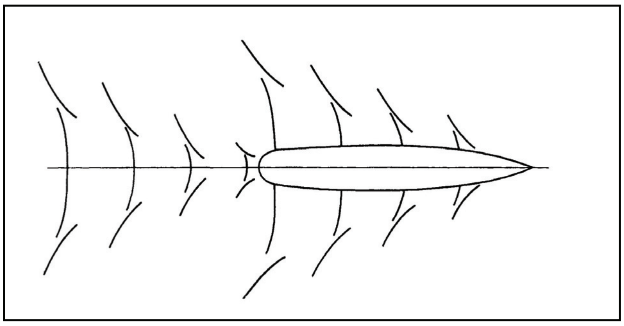

Figure 1: Schematic Diagram of Bow and Stern Wave Systems

Also shown in Figure 1 is the stern wave system which mimics that from the bow, although, in a real situation, this system will be affected by viscous wake and propulsor effects. It may also be noted in passing that a study of wave patterns, more detailed than the simple approach presented here, reveals a phase shift of 90^{o} between transverse and diverging waves at the cusp point.

Other wave systems arise from features of the hull design such as shoulders, or, more generally, from the rate of longitudinal and vertical changes of curvature of the hull geometry between bow and stern.

Shallow water, defined as water bounded in depth but laterally unbounded, also changes the wave pattern and this will be discussed in more detail below. Suffice it to say at this point that in shallow water wave amplitudes increase, as do the angles of the diverging waves to the sailing line.

4. HAVELOCK'S GRAPHICAL METHOD

Havelock's graphical method in [1] shows that a Kelvin Wave Pattern can be obtained for a point impulse, or disturbance, by the superposition of a number of plane wave systems rotated in azimuth in the (x,y) plane, as indicated in his Figures 1 and 2 in [1], reproduced here as Figure 2.

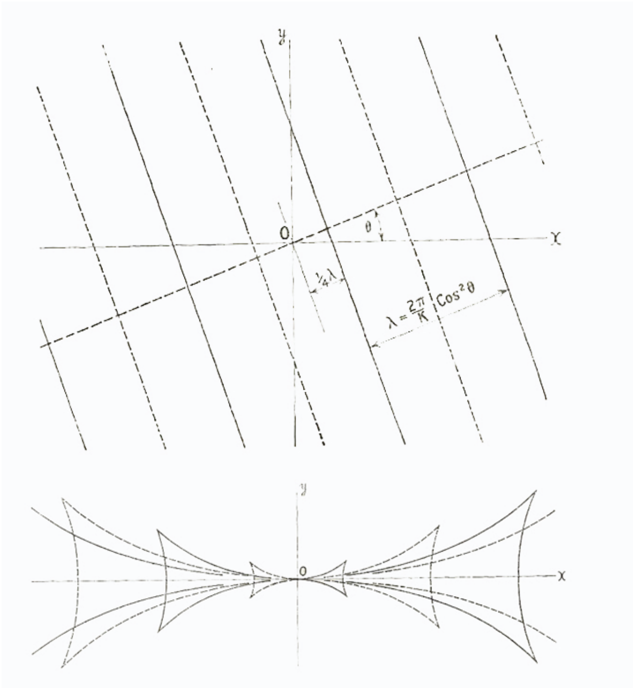

Figure 2: Havelock's Graphics from Reference 1

Trains of waves are shown by lines indicating crests and troughs in the upper part of Figure 2. They are assumed to be moving freely, each one at wave celerity c, derived from equation (1):

$$c = g\lambda.tanh(2\pi h/\lambda)/(2\pi))^{\frac{1}{2}} \quad (6)$$

In shallow water, as $2\pi h/\lambda$ reduces to a small value, $\tanh(2\pi h/\lambda)$ tends to $2\pi h/\lambda$ and wave celerity, c, tends to:

$$c = (gh)^{\frac{1}{2}} \quad (7)$$

This is an important result for wash studies in shallow water because it defines the critical wave celerity for a given depth of water, h. From this comes the critical Depth Froude Number, defined as $F_{nh} = V/(gh)^{\frac{1}{2}} = 1.0$. Values of F_{nh} less than this are termed sub-critical while those greater than this are termed super-critical. The so-called trans-critical region extends over the approximate range $0.8 \leq F_{nh} \leq 1.2$.

Havelock's graphical approach may seem counter-intuitive as an introduction to the forced waves of the Kelvin wave pattern, but Havelock assumed that the waves, travelling at an azimuth angle ß, had a velocity in the direction of travel of c.cosß with wavelengths adjusted accordingly. Considering all wave trains over a range of azimuth angles ß, he showed that this gave a pattern of waves all moving parallel to the x-axis at a speed c. He also asserted that, from this, patterns of waves of constant phase emerge – the Kelvin wave pattern.

As mentioned above, he indicated the method in the paper, but does not show the complete superposition of all wave trains in the construction and, furthermore, did not extend this graphical method to shallow water and trans-critical conditions.

To remedy this, a short computer program was written for this paper to carry out the superposition in deep and shallow water with the aim of taking the Depth Froude Number of the motion as close to unity as possible while still, of course, retaining the dispersive relationship. Similar work of more limited range is described in [3].

The software solved the dispersion equation (Equation (6)) for all $F_{nh} < 1.0$, using the Newton-Raphson method to determine the appropriate value of wave number k which was then used to deduce wavelength. Trains of plane wave crests and troughs with this wavelength, resolved into the direction of motion, were then plotted over a range of azimuth angles in the (x,y) plane from -ß^{o} to ß^{o} in increments of δß.

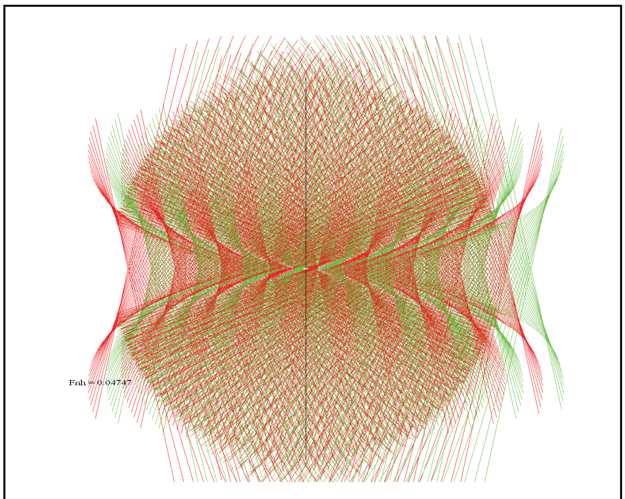

Figure 3: Wave Pattern in Deep Water

Deep water results for a point disturbance, including all construction lines, are shown in Figure 3; crests and troughs are shown, ß was set to 70^{o} to improve clarity and, initially, δß to 2^{o}.

The appearance of the familiar Kelvin wave pattern is apparent, and several points of interest may be noted in the plot:

- Transverse and diverging crest/trough lines are evident as the plane waves combine to form the familiar pattern.
- The angle of the cusp line is around 20^{o} as is to be expected from Kelvin's early work and many experiment measurements made since.
- The Kelvin wave pattern for a point pressure disturbance is symmetrical about x = 0. This is a feature of the algebraic results obtained by the pioneers of this work which they might remove by superimposing a dummy wave system to cancel out the offending waves, or use a fictitious friction factor set to zero on completion of the analysis.
- In the "circle" of construction lines there is a welter of crests and troughs implying, as Havelock suggests, that a good deal of wave cancelling may occur in this and other areas of the (x,y) plane. This remains to be demonstrated, however.
- The lines of equal phase which constitute the Kelvin pattern are in effect a series of hypocycloid curves, each having the same apex passing through a common point.

Although the plot in Figure 3 gives a recognisable representation of a ship wave system in deep water, it is only valid for a point disturbance. As already mentioned, in reality there would be a system such as this from the bow, a similar system from the stern and two or more from the curved part of the hull in between. It is the resolution of the mathematical or numerical difficulties in representing these hull shape effects on the wave pattern that demanded a great deal of work by pioneers prior to, and after, the coming of the digital computer. Modern numerical methods can now deal with these problems more readily, but accurate representation of shallow water effects, especially in the trans-critical region, is still difficult to deal with both analytically and numerically with some CFD techniques. However, the purpose of this paper is one of illustration and illumination, rather than prediction so the results for a point disturbance are adequate for what follows.

5. HAVELOCK'S METHOD EXTENDED

Although Havelock's graphical method shows ship-like wave systems forming from the superposition of plane wave trains, it has its limitations. These are:

- Only the crest and trough locations are given, with no information about wave shape between crest and trough. There is also no information in the third, vertical, dimension, the dimension in which wave amplitude is observed.

- Wave cancellation is assumed, but not demonstrated.
- No information on wave decay with distance off is given.

To overcome these shortcomings, the method was extended. In Havelock's day, the production, by hand, of plots such as those shown above would have been extremely time-consuming. Havelock had no modern computers at his disposal for the work described in [1] and was therefore unlikely to explore his graphical method further. It had served its purpose as a demonstration.

However, on a computer, it is a simple extension to the method to superimpose a square grid of points over a plot such as that shown in Figure 3 and compute the wave amplitude at each grid point for each wave train, summing these for each ß value.

Local wave amplitude, ζ, at each grid point, for each wave train at each ß can be computed from the well-known expression:

$$\zeta(x,y) = a \sin(k \cos(\beta).x + k \sin(\beta).y) \tag{8}$$

where time-dependency and phase angle have been ignored as irrelevant in this case. In Equation (8) "a" is the arbitrary amplitude for all wave trains and k is the wave number appropriate to the Depth Froude Number and ß value.

Accordingly, the original computer program was modified to extend the Havelock method in this way and three-dimensional perspective views of the resultant wave system were obtained.

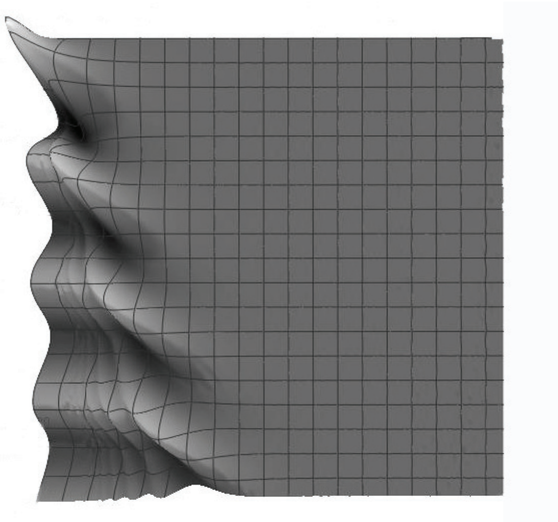

Figure 4: Computed Wave Pattern: Deep Water

Scanning from -80° to 80° and using a δß of 0.1°, the plot in Figure 4 resulted. This is for deep water and corresponds to the two-dimensional case in Figure 3. Only the first quadrant is shown, the line of symmetry being on the left with its origin at the top; the viewpoint chosen is from directly above the wave system. The extent of the grid in the x and y directions was limited to four deep water wavelengths and this scale was maintained for all subsequent plots for ease of comparison. The vertical scale was entirely arbitrary. For all plots the computation time for all amplitudes was around one second. Computation time for the perspective plot was a few seconds, so the production of the wave pattern plots was very rapid.

In Figure 4, it is seen that, as Havelock predicted, a good deal of wave cancellation does in fact take place, although there is some noise on the "calm" water brought about, no doubt, by matters of numerical resolution. Transverse and diverging waves can be seen in the wave system, wave height decay with distance off is clear and intermediate wave contours are reasonably well represented.

If speed is kept constant and water depth reduced, shallow water effects may be explored, defined in what follows by the Depth Froude Number, F_{nh}.

Results at a Depth Froude Number in the sub-critical region of 0.46 are shown in Figure 5.

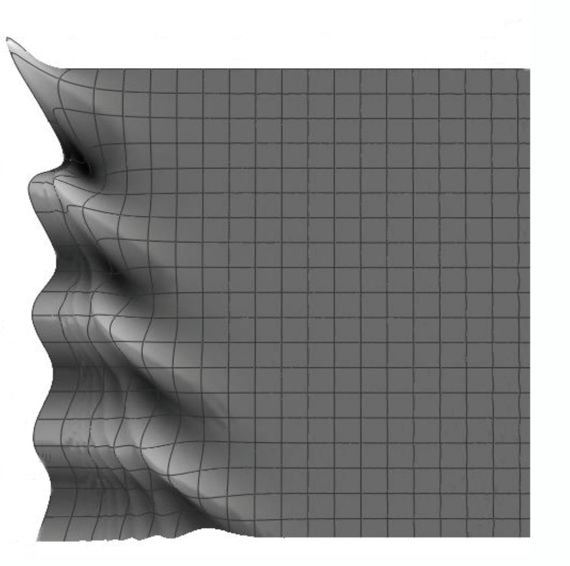

Figure 5: Computed Wave Pattern: Depth Froude Number = 0.46

In this Figure, the following points are apparent:

- The cusp line angle is slightly greater than the value of around 20° obtained in deep water (Figure 4).
- The transverse wavelengths are similar to, but slightly greater than, those in deep water
- The shape, typical of the Kelvin wave pattern, is, however, still clear and very similar to that obtained in deep water.

If now the Froude Number is increased to a value of 0.84, near the start of the trans-critical region, the results in Figure 6 are obtained.

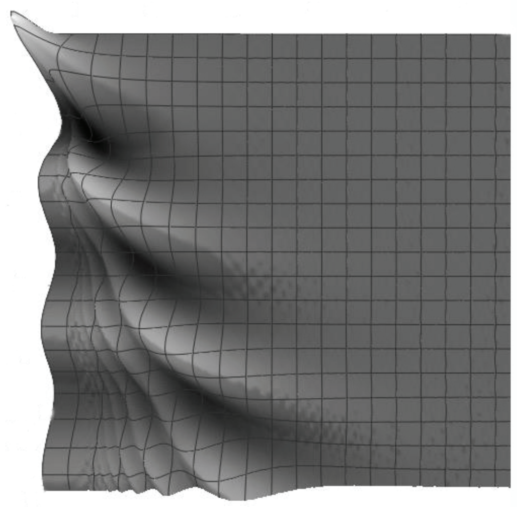

Figure 6: Computed Wave Pattern: Depth Froude Number = 0.84

It is immediately seen that:

- The transverse wavelengths have increased
- The angle between the sailing line and the cusp line has increased noticeably beyond the deep water value around 20^{o}.
- Bow wave height has increased.

Increasing the Depth Froude Number closer to the critical value, the resultant changes are shown in Figures 7 and 8.

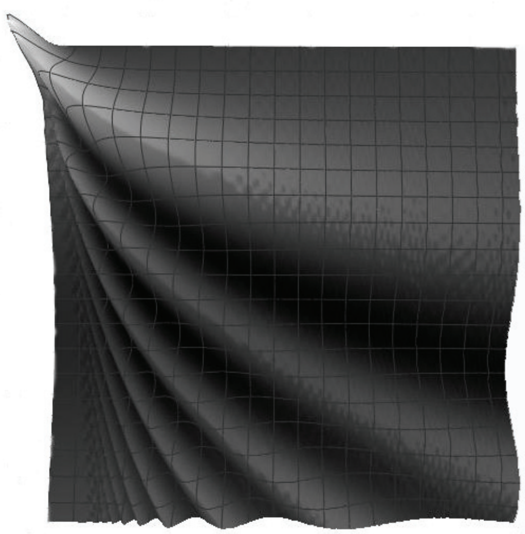

Figure 7: Computed Wave Pattern: Depth Froude Number = 0.99

Figure 7 corresponds to a sketch of two wave crests given by Havelock in Figure 9 of [4]. It is clear that when the Depth Froude Number is so close to unity, the transverse wavelength increases significantly to such an extent that transverse waves and the cusp line are, to all intents and purposes, disappearing, or have disappeared, from the area of interest. Diverging waves now dominate the scene with their crests convex to the sailing line at a very large angle. The first wave is dominant and may appear to be a precursor to a solitary wave or soliton. That this is not such a wave is discussed further below.

Increasing Depth Froude Number to 0.999 gives the results shown in Figure 8.

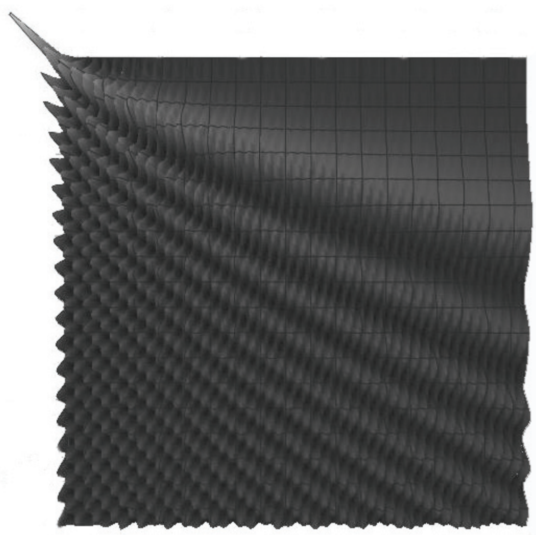

Figure 8: Computed Wave Pattern: Depth Froude Number = 0.999

Here it is seen that the transverse waves have effectively disappeared due to their extreme length. Information on wavelengths, resolved into the direction of motion for all ß has been extracted from the computer program and is shown in Figure 9 for Depth Froude Numbers of 0.46 and 0.9999.

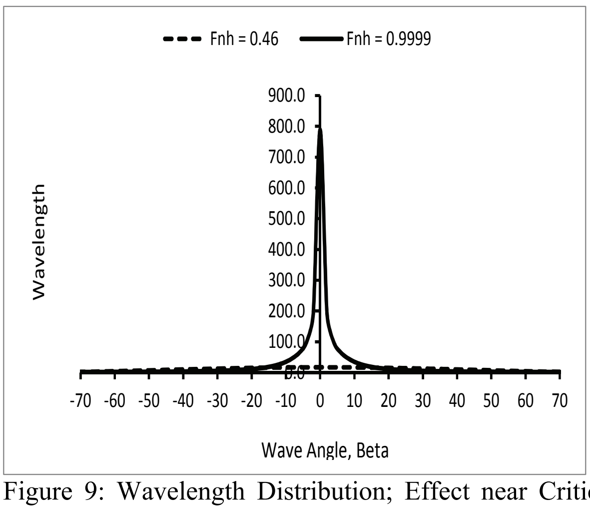

Figure 9: Wavelength Distribution; Effect near Critical Conditions

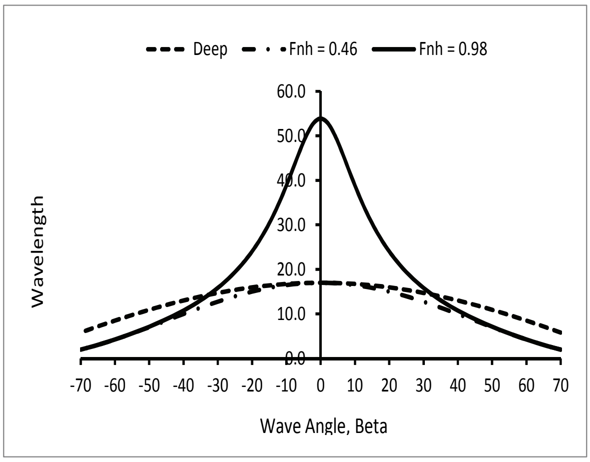

Figure 10: Wavelength Distribution up to Transcritical Conditions

The extreme length of the transverse wave on the sailing line (ß = 0^o) may be noted and confirms that, for this case, the transverse wave system has effectively disappeared. A further comparison is shown in Figure 10 for lower Depth Froude Numbers which confirms the trend and shows the lesser contributions to transverse waves at higher ß values in shallow water compared to deep.

6. WAVE HEIGHT DECAY

Wave height decay with lateral distance from the sailing line (either along the Kelvin cusp line or perpendicular to the sailing line) was explored by Havelock in [4] and has been used by others as a means of deducing wave amplitudes (see [5] for example). While Havelock, in [4], gives information for the decay exponent in deep water, he does not do so for shallow water. As wave wash is more pronounced in shallow water, it is not without interest to explore wave decay as predicted by the extended numerical Havelock method described above.

It is usual to express wave decay (see [5] for example) by the simple expression:

$$\zeta(\xi) = \gamma . \xi^n \tag{8}$$

where the distance ξ is usually along the wave crest, the cusp line or the sailing line or perpendicular to the sailing line.

For deep water, and for transverse waves *along the sailing line*, the exponent, n, from the numerical method is about -0.41, compared to Havelock's value of -0.5. For a Depth Froude Number of 0.9999, the decay exponent, *along the crest of the first wave*, was found to be -0.48. It should be stressed, however, that measurement of wave decay from the numerical results in these two cases was rather crude.

In passing, it may be mentioned that, in the trans-critical region, it has been shown experimentally by others that unsteady effects are likely to play a significant part. They arise from the fact that, as the critical Depth Froude Number is approached, the time required for the long transverse wave to be established approaches infinity, so steady conditions are not, for all practical purposes, achieved. This behaviour, and the possible generation of solitary waves at the same time (see below), will affect wave decay in the trans-critical region.

7. DISCUSSION

In this paper, the extended Havelock method, being numerically-based and hence more computer-friendly, has introduced a third dimension – wave amplitude – into the discussion and this, with the resulting graphics, has illuminated the changes in wave pattern as the critical Depth Froude Number is approached.

Perhaps of most interest is the effect on the transverse wave system. It is well-known, and indeed explained by Havelock in [4], that the transverse wave system can no longer exist in supercritical shallow water conditions. What is of interest in the present study is that, for all practical purposes, the transverse system plays no real part in the shallow water wave pattern slightly *before* the critical Depth Froude Number is reached. This may perhaps help to explain why "critical" conditions are sometimes found to exist - in measurements of resistance for example - before a Depth Froude Number of unity is reached.

It has also been shown that the diverging wave system, which dominates the wave pattern in the trans-critical phase, is characterised by a large-amplitude, substantial, leading wave which could be confused with the development of a solitary wave or "soliton". While this is an understandable confusion, it should be remembered that solitons are non-dispersive and therefore do not satisfy equation (1). They can therefore play no part in the sub-critical wave patterns discussed here. Solitons are high energy waves, tend to be characterised by a constant height crest with no trough, and can run for great distances without significant decay; their formation is not always guaranteed in shallow water, and depends, among other things, on water depth as well as speed (see [6]).

Finally, it is of interest to speculate on how the extended, numerical, method could be further developed. All the results presented in this paper have been for a single pressure disturbance moving in a straight line at a steady speed. It would be possible, presumably, to introduce further pressure points to form a distribution which, assuming the "hull" surface condition was satisfied, might give some idea of a body moving through the water. However, whether such an exercise would give more by way of illustration and understanding than the simple examples given here is open to question.

8. CONCLUSIONS

As a result of the study described above, the following conclusions may be drawn:

- Havelock's graphical method can be extended using numerical techniques to include the third dimension.
- The transverse wave system tends to play an increasingly minor, and ultimately negligible, role in the wave pattern as the critical Depth Froude Number is approached.
- Wave decay, as predicted by the numerical method, roughly conforms to Havelock's prediction.
- The numerical method provides a quick and convenient method of exploring vessel wave pattern characteristics in shallow water.
- Wave amplitude increases as Depth Froude Number approaches unity.
- Near the critical Depth Froude Number the wave pattern is dominated by a large initial wave crest which could be confused with a solitary wave. As a solitary wave is non-dispersive, this cannot be the case, although such waves may form independently.

9. REFERENCES

1. HAVELOCK, T H, Wave Patterns and Wave Resistance, *Trans RINA, Volume LXXVI, 1934, pp 430 to 446*
2. LEWIS, E V (ed), Principles of Naval Architecture. Volume II: Resistance, Propulsion and Vibration, *SNAME, Jersey City, 1988*
3. LEE C, LEE B W, KIM Y J, KO K O, Ship Wave Crests in Intermediate-Depth Water, *Proceedings of Sixth International Conference on Asian and Pacific Coasts (APAC 2011), December 2011, Hong Kong*
4. HAVELOCK, T H, The Propagation of Groups of Waves in Dispersive Media, with Application to Waves in Water produced by a Travelling Disturbance, *Proceedings of the Royal Society, A, Vol 81, 1908*
5. ROBBINS, A AND RENILSON, M R, A Tool for the Prediction of Wave Wake, *Transactions of RINA, International Journal of Maritime Engineering, part A1, 2006*
6. ROBBINS, A, THOMAS, G, DAND, I, MACFARLANE, G AND RENILSON, M, When is Water Shallow?, *Transactions of RINA, International Journal of Maritime Engineering, Vol 155 part A3, 2013*