STRENGTH OF SHIPS' GRILLAGES UNDER LATERAL LOAD AND IN-PLANE COMPRESSION

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A Z Lokshin, Formerly Professor at St. Petersburg State Maritime Technical University, Russia, US, **V G Mishkevich**, AV Streamline, US, and **L D Ivanov** retired, formerly with American Bureau of Shipping, Houston, US

SUMMARY

The paper deals with strength of a grillage loaded by lateral load and in-plane compression load (in one direction). It consists of a system of prismatic girders crossing under 90°. The compression load is taken by the longitudinal girders that are elastically fixed on rigid supports. The system of aggregated differential equations is derived for solution of the problem using the Lagrange method. It allows for replacement of the system of aggregated differential equations by a system of independent differential equations. These equations for the case of simultaneous action of lateral and longitudinal compression load have the form of differential equations for bending of prismatic girders laying on elastic foundation and loaded with lateral and longitudinal compression forces. When only lateral load exists, the form of these equations coincides with the form of differential equations for bending of girders laying on elastic foundation and loaded with lateral load alone. When only longitudinal compression load exists, the form of these equations coincides with the form of differential equations for buckling of girders laying on elastic foundation.

Solutions are given for bending of a grillage (the first two problems). Formulas are derived for calculation of the parameters of longitudinal girders' bending when girders' end sections are elastically fixed. Also, formulas are derived for calculation of the reaction forces at cross-points of transverse and longitudinal girders. When only longitudinal compression load exists (third problem), a solution is given for the connection between the coefficient of elastic foundation's rigidity and the Euler force. Results obtained by using the proposed method are compared with FEA simulations.

NOMENCLATURE

a	distance between transverse girders
b	distance between longitudinal girders
E	modulus of elasticity
f	girder's cross-sectional area
i_0	moment of inertia of transverse girder
J or J_0	moment of inertia of longitudinal girder
k_m	coefficient of rigidity of the elastic foundation
L	length of longitudinal girder
l	length of transverse girder
n	number of longitudinal girders
$P_m(x)$	main deflection function
Q	lateral load on a transverse girder
q	distributed lateral load on the grillage
q_m	distributed lateral load on longitudinal girder
q_m^0	partial solution of the differential equation of longitudinal girder's bending
R	reaction force
T	in-plane compression force acting on a longitudinal girder
T_E	Euler force
$w_i(x)$	longitudinal girder's deflection curve
β_i	coefficient representing the influence of the load Q(x) on the deflection of i^{th} joint of the transverse girder
γ_{ij}	coefficient representing the influence of the reaction force of j^{th} longitudinal on the deflection of i^{th} joint of the transverse girder
v_{jm}	forms of the main deflection
α	coefficient of pliability of elastic fixity
η_m	parameter depending on T and k_m
λ_m	parameter depending on k_m

1. INTRODUCTION

The object of this study is the strength of a grillage built with orthogonally crossing prismatic beams covered by plating. The beams parallel to axis OX (Figure 1) have the same moments of inertia and the same boundary conditions (the boundary conditions can be any). They are labeled "longitudinals" loaded with a compressive force. The beams transverse to longitudinals (labeled "transverses") also have the same moments of inertia and the same boundary conditions (the boundary conditions can be any). The paper deals with grillages with a large number of equally distant transverses (sometimes they are labeled "main direction beams").

The theory of grillages with longitudinals having different moments of inertia loaded only with lateral load was developed first by Bubnov I G (1912). He derived a system of differential equations for bending of such types of grillages and proposed a method for solution of these equations. The calculations following his method were complicated and the method was not used in practice. The following assumptions are basis of these differential equations:

- The lateral load is carried only by the transverses.
- The direction of the reaction force R_j in the crossing points of longitudinals and transverses is perpendicular to the grillage plane.

- The reaction force R_j acting on longitudinals (resulting from the interaction between any transverse and j^{th} longitudinal) can be replaced by evenly distributed load r_j, i.e.,

$$r_j(x) = \frac{R_j(x)}{a} \tag{1}$$

where a = distance between forces $R_j(x)$.

The possibility of using the above-mentioned assumptions is analyzed by Papkovich (1947).

2. DERIVATION OF THE DIFFERENTIAL EQUATIONS FOR BENDING OF A GRILLAGE

The derivation of the differential equations for bending of a grillage first requires determination of the deflection of the transverses at the crossing point with the i^{th} longitudinal, i.e.,

$$w_i(x) = \beta_i \frac{Q(x) l^3}{E i_0} - \frac{l^3}{E i_0} \sum_{j=1}^{n} R_j(x) \gamma_{ij} \tag{2}$$

where

$w_i(x)$ = deflection of any transverse at section of crossing with i^{th} longitudinal;

$Q(x)$ = load on a transverse located at a distance x from the origin of the coordinate system (Figure 1);

i_0 = moment of inertia of the transverse;

l = length of the transverse;

$R_j(x)$ = reaction force resulting from the interaction between any transverse and j^{th} longitudinal;

γ_{ij} = coefficient representing the influence of the reaction force of j^{th} longitudinal on the deflection of i^{th} joint of the transverse;

β_i = coefficient representing the influence of the load Q(x) on the deflection of i^{th} joint of the transverse;

n = number of longitudinals.

The assumption in Eq. 2 is that the longitudinals support the transverses. It can be written for each longitudinal. The deflection of j^{th} longitudinal at a cross section with any transverse is equal to the deflection of the transverse at the same section. For a large number of transverses, one can consider the deflection w_j as a continuous function of x. Then,

$$a E J_j w_j^{IV}(x) = R_j(x) \tag{3}$$

where J_j = moment of inertia of j^{th} longitudinal.

By substituting R_j from Eq. (3) in equality (2), one can find

$$w_i(x) = \beta_i \frac{Q(x) l^3}{E i_0} - \frac{a l^3}{E i_0} \sum_{j=1}^{n} E J_j w_j^{IV}(x) \gamma_{ij} \tag{4}$$

The system of differential equations (4) was derived by Bubnov I G (1912). A simple solution of system (4) was proposed by Papkovich P F (1947). The proposed method was labeled "method of main deflections". The essence of the method is in the possibility of replacing the system of differential equations (4) by independent differential equations. Several quadratic polynomials are included in the method which makes the method too cumbersome. Korotkin, Lokshin and Sivers (1953) proposed an even simpler method which is used in this paper for calculation of the strength of a grillage loaded by in-plane and lateral loads.

The system of differential equations for a grillage with equal longitudinals loaded by the two loads can be derived if in Eq. (2) the following equation is used (instead of Eq. (3):

$$a\left[E J w_j^{IV}(x) + T w_j^{II}(x)\right] = R_j(x) \tag{5}$$

where T = in-plane compressive force acting on a longitudinal; J = moment of inertia of longitudinals.

As a result, the following system of equations is obtained:

$$\left.\begin{aligned} & w_i(x) = \beta_i \frac{Q(x) l^3}{E i_0} - \frac{a l^3}{E i_0} \sum_{j=1}^{n} \gamma_{ij} \left\{ E J w_j^{IV}(x) + \right. \\ & \left. + T w_j^{II}(x) \right\} \end{aligned}\right\} \tag{6}$$

For solution of Eq. (6), Lagrange substitution is used, which leads to the following formula for the case with equal longitudinals.

$$w_j(x) = \frac{J_0}{J} \sum_{m=1}^{n} \nu_{jm} P_m(x) \tag{7}$$

where J_0 = any constant with dimensions for moment of inertia; ν_{jm} = yet unknown coefficients; $P_m(x)$ = also yet unknown function.

By substituting Equality (7) in Eq. (6) for $J_0 = J$, one can obtain:

$$\left.\begin{aligned} & \sum_{m=1}^{n} \nu_{im} P_m(x) = \beta_i \frac{Q(x) l^3}{E i_0} - \\ & - \frac{a l^3}{E i_0} \sum_{j=1}^{n} \gamma_{ij} \sum_{m=1}^{n} \left[E J_0 P_m^{IV}(x) + T P_m^{II}(x) \right] \nu_{jm} \end{aligned}\right\} \tag{8}$$

Since the function $P_m(x)$ is unknown, one can consider it as the solution of the differential equation of beams subjected to in-plane and lateral load laying on elastic foundation, i.e.,

$$E J_0 P_m^{IV}(x) + T P_m^{II}(x) = -k_m P_m(x) + q_m(x) \tag{9}$$

where

k_m = coefficient of rigidity of the elastic foundation to be determined;

$q_m(x)$ = distributed lateral load to be determined as well.

The functions $P_m(x)$ were labeled by Papkovich (1947) "main deflections" and the coefficients ν_{jm} – "forms of main deflections".

Taking into consideration Equality (9), the system of differential equations (8) will take the following form:

$$\left.\begin{aligned}&\sum_{m=1}^{n}\nu_{im}P_m(x)=\beta_i\frac{Q(x)l^3}{Ei_0}-\\&-\frac{al^3}{Ei_0}\sum_{j=1}^{n}\gamma_{ij}\sum_{m=1}^{n}\nu_{jm}\left[q_m(x)-k_mP_m(x)\right]\end{aligned}\right\}\tag{10}$$

or

$$\left.\begin{aligned}&\sum_{m=1}^{n}\left[\nu_{im}-\frac{al^3k_m}{Ei_0}\sum_{j=1}^{n}\gamma_{ij}\nu_{jm}\right]P_m(x)=\\&=\beta_i\frac{Q(x)l^3}{Ei_0}-\frac{al^3}{Ei_0}\sum_{j=1}^{n}\gamma_{ij}\sum_{m=1}^{n}\nu_{jm}q_m(x)\end{aligned}\right\}\tag{11}$$

Since the coefficients ν_{jm} are unknown, one can determine them in such a way that for each index m the following equality is fulfilled

$$\nu_{im}-\frac{al^3k_m}{Ei_0}\sum_{j=1}^{n}\gamma_{ij}\nu_{jm}=0\tag{12}$$

The following parameters are introduced:

$$\lambda_m=\frac{Ei_0}{al^3k_m}\qquad k_m=\frac{Ei_0}{al^3\lambda_m}\tag{13}$$

Then

$$\nu_{im}\lambda_m=\sum_{j=1}^{n}\gamma_{ij}\nu_{jm}\tag{14}$$

The system of equations (14) can be presented in the following way:

$$\left.\begin{aligned}&i=1\\&(\gamma_{11}-\lambda_m)\nu_{1m}+\gamma_{12}\nu_{2m}+\gamma_{13}\nu_{3m}+\\&\qquad....+\gamma_{1n}\nu_{nm}=0\\&i=2\\&\gamma_{21}\nu_{1m}+(\gamma_{22}-\lambda_m)\nu_{2m}+\gamma_{23}\nu_{3m}+\\&\qquad....+\gamma_{2n}\nu_{nm}=0\\&\dots\dots\dots\dots\dots\dots\dots\dots\\&i=n\\&\gamma_{n1}\nu_{1m}+\gamma_{n2}\nu_{2m}+\gamma_{n3}\nu_{3m}+\\&\qquad...+(\gamma_{nn}-\lambda_m)\nu_{nm}=0\end{aligned}\right|\tag{15}$$

Equations (15) represent a system of linear homogeneous algebraic equations. It may have a non-trivial solution if the determinant is equal to zero, i.e.,

$$\begin{vmatrix}\gamma_{11}-\lambda_m & \gamma_{12} & \gamma_{13}\cdots\cdots & \gamma_{1n}\\ \gamma_{21} & \gamma_{22}-\lambda_m & \gamma_{23}\cdots\cdots & \gamma_{2n}\\ \dots & \dots & \dots & \dots\\ \gamma_{n1} & \gamma_{n2} & \gamma_{n3} & \gamma_{nn}-\lambda_m\end{vmatrix}=0\tag{16}$$

The coefficients γ_{ij} satisfy the equality

$$\gamma_{ij}=\gamma_{ji}\tag{17}$$

The determinant (16) has "n" real roots, i.e., "n" real values of the coefficient λ_m and, consequently, "n" values of the coefficient of rigidity of the elastic foundation k_m which are calculated by Eq. (13).

The forms of main deflections ν_{jm} can be calculated with accuracy up to an arbitrary co-multiplier. They satisfy the condition called "orthogonally condition" (Korotkin, Lokshin, Sivers, 1953). For equal longitudinal girders, this condition is:

$$\sum_{i=1}^{n}\nu_{im}\nu_{ir}=0\quad\text{for}\quad r\neq m\qquad\sum_{i=1}^{n}\nu_{ir}^2\neq0\tag{18}$$

Next is the determination of the lateral distributed load $q_m(x)$ in Eq. (10). Using Eq. (11) and considering Equalities (10) and (14), one can derive the following equality:

$$\left.\begin{aligned}&\beta_i\frac{Q(x)}{a}=\sum_{m=1}^{n}q_m(x)\sum_{j=1}^{n}\gamma_{ij}\nu_{jm}=\\&=\sum_{m=1}^{n}q_m(x)\nu_{im}\lambda_m\end{aligned}\right\}\tag{19}$$

To determine function $q_m(x)$, the two sides of Eq. (19) are multiplied by ν_{ir} and the results for each i are summed up. Thus,

$$\sum_{i=1}^{n}\beta_i\frac{Q(x)}{a}\nu_{ir}=\sum_{m=1}^{n}q_m(x)\lambda_m\sum_{i=1}^{n}\nu_{im}\nu_{ir}\tag{20}$$

Based on orthogonallity condition (18) and assuming r = m, one can obtain:

$$q_m(x)=\frac{Q(x)}{a\lambda_m}\frac{\sum_{i=1}^{n}\beta_i\nu_{im}}{\sum_{i=1}^{n}\nu_{im}^2}\tag{21}$$

To solve the differential equation (9), one should determine the boundary conditions of the function $P_m(x)$. Using the Equality (7), one can represent the function $P_m(x)$ by using the function $w_j(x)$. This is done by multiplication of the two sides of Equality (7) with ν_{ir} and by summation of the results for all values of the index j, i.e.,

$$\sum_{j=1}^{n} w_j(x)\nu_{jr} = \sum_{m=1}^{n} P_m(x)\sum_{j=1}^{n}\nu_{jm}\nu_{jr} \tag{22}$$

Bearing in mind that the forms of main deflections satisfy the condition of orthogonallity (18), one can obtain (for r = m and j = i) the following equation:

$$P_m(x) = \frac{\sum_{j=1}^{n} w_j(x)\nu_{jm}}{\sum_{j=1}^{n}\nu_{jm}^2} \tag{23}$$

Differentiating Eq. (23), one can obtain:

$$\left.\begin{aligned} P_m^{I}(x) &= \frac{\sum_{j=1}^{n} w_j^{I}(x)\nu_{jm}}{\sum_{j=1}^{n}\nu_{jm}^2} \\ P_m^{II}(x) &= \frac{\sum_{j=1}^{n} w_j^{II}(x)\nu_{jm}}{\sum_{j=1}^{n}\nu_{jm}^2} \\ P_m^{III}(x) &= \frac{\sum_{j=1}^{n} w_j^{III}(x)\nu_{jm}}{\sum_{j=1}^{n}\nu_{jm}^2} \end{aligned}\right\} \tag{24}$$

Particular cases:

a) All longitudinals are freely supported by rigid supports at x = 0 and x = *l* (See Figure 1). In this case:

$$w_j(0) = w_j(l) = 0; \quad w_j^{II}(0) = w_j^{II}(l) = 0 \tag{25}$$

Hence,

$$P_m(0) = P_m(l) = 0; \quad P_m^{II}(0) = P_m^{II}(l) = 0 \tag{26}$$

b) All longitudinals are elastically fixed and supported by rigid supports at x = 0 and x = *l* . In this case:

$$\left.\begin{aligned} w_j(0) &= w_j(l) = 0 \\ w_j^{I}(0) &= \alpha E J_0 w_j^{II}(0) \\ w_j^{I}(l) &= -\alpha_1 E J_0 w_j^{II}(l) \end{aligned}\right\} \tag{27}$$

Hence,

$$\left.\begin{aligned} P_m(0) &= P_m(l) = 0 \\ P_m^{I}(0) &= \alpha E J_0 P_m^{II}(0) \\ P_m^{I}(l) &= -\alpha_1 E J_0 P_m^{II}(l) \end{aligned}\right\} \tag{28}$$

where α and α_1 are coefficients of pliability of the elastic fixity.

3. INTEGRATION OF THE HOMOGENEOUS EQUATIONS

It follows from the above-given equations that the calculation of longitudinals' deflection $w_j(x)$ requires finding the function $P_m(x)$ that satisfies differential equation (9). The integral of this equation is equal to the sum of the solutions of a homogeneous equation (when $q_m(x) = 0$) and a partial solution which depends on the type of function $q_m(x)$.

The integral of the homogeneous equation can be found by Euler's method, i.e.,

Assume that

$$P_m(x) = A_m e^{S_m x} \tag{29}$$

where A_m is a constant. By substituting the functions $P_m(x)$ in Eq. (9) with Eq. (29) and putting q(x) = 0, one can find the following characteristic equation which determines the parameters S_m:

$$S_m^4 + \frac{T}{EJ_0}S_m^2 + \frac{k_m}{EJ_0} = 0 \tag{30}$$

The four roots of Eq. (30) are:

$$S_m = \pm\sqrt{\frac{T}{2EJ_0}}\sqrt{-1 \pm \sqrt{1 - \frac{4k_m EJ_0}{T^2}}} \tag{31}$$

By introducing the notations

$$\eta_m^2 = \frac{4k_m EJ_0}{T^2} \quad \text{and} \quad \gamma = \sqrt{\frac{T}{2EJ_0}} \tag{32}$$

One can represent Eq. (31) in the following form:

$$S_m = \pm\gamma\sqrt{-1 \pm \sqrt{1 - \eta_m^2}} \tag{33}$$

Thus, the general integral of the homogeneous equation can be rewritten in the following form:

$$P_m(x) = \sum_{j=1}^{4} A_m^{(j)} e^{S_m^{(j)} x} \tag{34}$$

The roots of the characteristic equation and, consequently, the type of the function $P_m(x)$ depend on the parameter η_m^2. Several cases are possible:

a) $\eta_m^2 = 1$.

In this case, all four roots of the characteristic equation will be imaginary and equal to

$$S_m^{(1,2)} = \pm i\gamma \tag{35}$$

Using the equations

$$\left.\begin{aligned} \sin(\gamma x) &= \frac{e^{i\gamma x} - e^{-i\gamma x}}{2i} \\ \cos(\gamma x) &= \frac{e^{i\gamma x} + e^{-i\gamma x}}{2} \end{aligned}\right\} \tag{36}$$

the general integral of the homogeneous equation will have the following form:

$$\left.\begin{aligned} P_m(x) = C_0\cos(\gamma x) + C_1\gamma x\cos(\gamma x) + \\ C_2\gamma x\sin(\gamma x) + C_3\sin(\gamma x) \end{aligned}\right\} \tag{37}$$

b) $\eta_m^2 < 1$.

The roots are imaginary and different:

$$S_m^{(1,2)} = \pm i\alpha_1 \qquad S_m^{(3,4)} = \pm i\alpha_2 \tag{38}$$

$$\left.\begin{aligned} \alpha_1 &= \gamma\sqrt{1 - \sqrt{1 - \eta_m^2}} \\ \alpha_2 &= \gamma\sqrt{1 + \sqrt{1 - \eta_m^2}} \end{aligned}\right\} \tag{39}$$

The general integral of the homogeneous equation will have the following form:

$$\left.\begin{aligned} P_m(x) = C_0^{/}\cos(\alpha_1 x) + C_1^{/}\sin(\alpha_1 x) + \\ C_2^{/}\cos(\alpha_2 x) + C_3^{/}\sin(\alpha_2 x) \end{aligned}\right\} \tag{40}$$

c) $\eta_m^2 > 1$.

In this case the roots will be complex ones

$$S_m^{(j)} = \pm(\alpha + i\beta) \qquad j = 1, 2, 3, 4 \tag{41}$$

$$\alpha = \frac{\gamma}{\sqrt{2}}\sqrt{\eta_m - 1} \qquad \beta = \frac{\gamma}{\sqrt{2}}\sqrt{\eta_m + 1} \tag{42}$$

The general integral of the homogeneous equation can be presented in the following way:

$$\left.\begin{aligned} &P_m(x) = C_0^{//}\mathrm{ch}(\alpha x)\cos(\beta x) + \\ &C_1^{//}\mathrm{ch}(\alpha x)\sin(\beta x) + C_2^{//}\mathrm{sh}(\alpha x)\sin(\beta x) + \\ &C_3^{//}\mathrm{sh}(\alpha x)\cos(\beta x) \end{aligned}\right\} \tag{43}$$

The general integral of Eq. (9) depends on the type of the lateral load $q_m(x)$. In the case when the lateral load on each transverse girder is the same (i.e., Q(x) = Q), Eq. (21) can be represented in the following form:

$$q_m = \frac{Q\sum_{i=1}^{n}\beta_i\nu_{im}}{a\lambda_m\sum_{i=1}^{n}\nu_{im}^2} \tag{44}$$

Further, considering Eq. (13), one can rewrite Eq. (44) in the following form:

$$q_m = \frac{Ql^3 k_m}{Ei_0}\frac{\sum_{i=1}^{n}\beta_i\nu_{im}}{\sum_{i=1}^{n}\nu_{im}^2} \tag{45}$$

4. PARTIAL SOLUTION OF THE EQUATION FOR BENDING OF A BEAM LOADED BY IN-PLANE COMPRESSION AND LATERAL LOAD

When the lateral load is determined by Eq. (45), the partial solution of Eq. (9) has the following form:

$$q_m^0 = \frac{q_m}{k_m} = \frac{Ql^3}{Ei_0}\frac{\sum_{i=1}^{n}\beta_i\nu_{im}}{\sum_{i=1}^{n}\nu_{im}^2} \tag{46}$$

This partial solution should be added to the solutions of the homogeneous equations, determined by Equalities (37), (40), (43).

5. DETERMINATION OF THE MAIN DEFLECTIONS

Let us determine the main deflections (i.e., functions $P_m(x)$) for a beam elastically fixed at rigid supports when $\eta_m^2 < 1$. In this case, the function $P_m(x)$ can be rewritten in the following form:

$$\left.\begin{aligned} P_m(x) = C_0^{/}\cos(\alpha_1 x) + C_1^{/}\sin(\alpha_1 x) + \\ C_2^{/}\cos(\alpha_2 x) + C_3^{/}\sin(\alpha_2 x) + q_m^{o} \end{aligned}\right\} \tag{47}$$

When the lateral load is evenly distributed, the function $P_m(x)$ is symmetrical relative to the middle section. Assume the origin of the coordinate system at this middle section (Figure 2). For symmetric bending, the coefficients of the odd functions in Equality (47) should be equal to zero. Then, the function $P_m(x)$ is

$$P_m(x) = C_0^{/}\cos(\alpha_1 x) + C_2^{/}\cos(\alpha_2 x) + q_m^o \tag{48}$$

For this case, the boundary conditions can be presented in the following form:

$$\left.\begin{array}{l} \text{For } x = \dfrac{L}{2}: \\ P_m\left(\dfrac{L}{2}\right) = 0 \quad P_m^{/}\left(\dfrac{L}{2}\right) = -\alpha E J_0 P_m^{//}\left(\dfrac{L}{2}\right) \end{array}\right\} \tag{49}$$

where α is the coefficient of pliability of the elastic fixity. The following coefficient is introduced:

$$\zeta = \frac{1}{1 + \dfrac{2\alpha E J_0}{L}} \tag{50}$$

Thus,

$$\alpha = \frac{L}{2 E J_0}\frac{(1-\zeta)}{\zeta} \tag{51}$$

Then, the second boundary condition in Eq. (49) takes the following form:

$$\left.\begin{array}{l} P_m\left(\dfrac{L}{2}\right) = 0 \\ P_m^{/}\left(\dfrac{L}{2}\right) = -\dfrac{L}{2}\dfrac{(1-\zeta)}{\zeta}P_m^{//}\left(\dfrac{L}{2}\right) \end{array}\right\} \tag{52}$$

By applying boundary conditions (52) to Equality (48) one can obtain the following system of equations

$$\left.\begin{array}{l} C_0^{/}\cos\nu_1 + C_2^{/}\cos\nu_2 = -q_m^o \\ C_0^{/}\left[\alpha_1\sin\nu_1 + \dfrac{L}{2}\dfrac{(1-\zeta)}{\zeta}\alpha_1^2\cos\nu_1\right] + \\ C_2^{/}\left[\alpha_2\sin\nu_2 + \dfrac{L}{2}\dfrac{(1-\zeta)}{\zeta}\alpha_2^2\cos\nu_2\right] = 0 \end{array}\right\} \tag{53}$$

$$\nu_1 = \frac{L}{2}\alpha_1 \qquad \nu_2 = \frac{L}{2}\alpha_2 \tag{54}$$

The solution of this system is:

$$C_0^{/} = -q_m^o \frac{\nu_2\sin\nu_2 + \dfrac{(1-\zeta)}{\zeta}\nu_2^2\cos\nu_2}{A+B} \tag{55}$$

$$C_2^{/} = q_m^o \frac{\nu_1\sin\nu_1 + \dfrac{(1-\zeta)}{\zeta}\nu_1^2\cos\nu_1}{A+B} \tag{56}$$

$$\left.\begin{array}{l} A = \nu_2\cos\nu_1\sin\nu_2 - \nu_1\cos\nu_2\sin\nu_1 \\ B = \dfrac{(1-\zeta)}{\zeta}\left(\nu_2^2 - \nu_1^2\right)\cos\nu_2\cos\nu_1 \end{array}\right\} \tag{57}$$

Similar calculations can be carried out for $\eta_m^2 > 1$.

For these values of η_m^2, the functions $P_m(x)$ (taking into consideration the symmetric bending of the beam), will be equal to

$$\left.\begin{array}{l} P_m(x) = C_0^{//}\text{ch}(\alpha x)\cos(\beta x) + \\ \qquad C_2^{//}\text{sh}(\alpha x)\sin(\beta x) + q_m^0 \end{array}\right\} \tag{58}$$

It follows that

$$\left.\begin{array}{l} P_m^{/}(x) = C_0^{//}\left[\alpha\text{sh}(\alpha x)\cos(\beta x) - \beta\,\text{ch}(\alpha x)\sin(\beta x)\right] + \\ \quad + C_2^{//}\left[\alpha\text{ch}(\alpha x)\sin(\beta x) + \beta\,\text{sh}(\alpha x)\cos(\beta x)\right] \end{array}\right\} \tag{59}$$

$$P_m^{//}(x) = \frac{4}{L^2}\left[C_0^{//}f(x) + C_2^{//}g(x)\right] \tag{60}$$

$$\left.\begin{array}{l} f(x) = \left(\bar{\nu}_1^2 - \bar{\nu}_2^2\right)\text{ch}(\alpha x)\cos(\beta x) - \\ \quad - 2\nu_1\nu_2\,\text{sh}(\alpha x)\sin(\beta x) \end{array}\right\} \tag{61}$$

$$\left.\begin{array}{l} g(x) = \left(\bar{\nu}_1^2 - \bar{\nu}_2^2\right)\text{sh}(\alpha x)\sin(\beta x) + \\ \quad + 2\nu_1\nu_2\,\text{ch}(\alpha x)\cos(\beta x) \end{array}\right\} \tag{62}$$

$$\bar{\nu}_1 = \frac{L}{2}\alpha \qquad \bar{\nu}_2 = \frac{L}{2}\beta \tag{63}$$

Using boundary conditions (27) and (28), one can derive the following system of equations determining the two coefficients $C_0^{//}$ and $C_2^{//}$

$$\left.\begin{array}{l} C_0^{//}\text{ch}(\bar{\nu}_1)\cos(\bar{\nu}_2) + C_2^{//}\text{sh}(\bar{\nu}_1)\sin(\bar{\nu}_2) = -q_m^0 \\ C_0^{//}\left[a_1 + \dfrac{1-\zeta}{\zeta}f\left(\dfrac{L}{2}\right)\right] + C_2^{//}\left[b_1 + \dfrac{1-\zeta}{\zeta}g\left(\dfrac{L}{2}\right)\right] = 0 \end{array}\right\} \tag{64}$$

where

$$\left.\begin{array}{l} a_1 = \bar{\nu}_1\,\text{sh}\,\bar{\nu}_1\cos\bar{\nu}_2 - \bar{\nu}_2\,\text{ch}\,\bar{\nu}_1\sin\bar{\nu}_2 \\ b_1 = \bar{\nu}_1\,\text{ch}\,\bar{\nu}_1\sin\bar{\nu}_2 + \bar{\nu}_2\text{sh}\,\bar{\nu}_1\cos\bar{\nu}_2 \end{array}\right\} \tag{65}$$

$$\left.\begin{aligned} f\left(\frac{L}{2}\right) &= \left(\bar{v}_1^{\,2} - \bar{v}_2^{\,2}\right) ch\bar{v}_1 \cos\bar{v}_2 - 2\bar{v}_1\bar{v}_2\, sh\bar{v}_1 \sin\bar{v}_2 \\ g\left(\frac{L}{2}\right) &= \left(\bar{v}_1^{\,2} - \bar{v}_2^{\,2}\right) sh\bar{v}_1 \sin\bar{v}_2 + 2\bar{v}_1\bar{v}_2\, ch\bar{v}_1 \cos\bar{v}_2 \end{aligned}\right\} \quad (66)$$

The solution of the system of equations leads to

$$C_0^{//} = -q_m^o \frac{b_1 + \frac{1-\zeta}{\zeta} g\left(\frac{L}{2}\right)}{\Delta_2} \quad (67)$$

$$C_2^{//} = q_m^o \frac{a_1 + \frac{1-\zeta}{\zeta} f\left(\frac{L}{2}\right)}{\Delta_2} \quad (68)$$

$$\left.\begin{aligned} \Delta_2 &= \bar{v}_1 \sin\bar{v}_2 \cos\bar{v}_2 + \bar{v}_2 ch\bar{v}_1 sh\bar{v}_1 + \\ &+ \frac{1-\zeta}{\zeta} 2\bar{v}_1\bar{v}_2 \left[\left(ch\bar{v}_1 \cos\bar{v}_2\right)^2 + \left(sh\bar{v}_1 \sin\bar{v}_2\right)^2\right] \end{aligned}\right\} \quad (69)$$

Substituting the values of the constants $C_0^{//}$ and $C_2^{//}$ in Eq. (58) and Eq. (60), one can derive the following formulas:

$$P_m(x) = -q_m^o \frac{1}{\Delta_2}\left\{b_1 ch(\alpha x)\cos(\beta x) - a_1 sh(\alpha x)\sin(\beta x) + \right.$$
$$\left. + \frac{1-\zeta}{\zeta}\left[g\left(\frac{L}{2}\right) ch(\alpha x)\cos(\beta x) - f\left(\frac{L}{2}\right) sh(\alpha x)\sin(\beta x)\right]\right\} + q_m^o \quad (70)$$

$$\begin{aligned} P_m^{//}(x) = &-\frac{4}{L^2} q_m^o \frac{1}{\Delta_2}\left\{b_1 f(x) - a_1 g(x) + \right. \\ &\left. + \frac{1-\zeta}{\zeta}\left[g\left(\frac{L}{2}\right) f(x) - f\left(\frac{L}{2}\right) g(x)\right]\right\} \end{aligned} \quad (71)$$

6. SOLUTIONS FOR EVEN NUMBER OF LONGITUDINAL GIRDERS

The calculation of the buckling strength is simplified when the longitudinals are located symmetrically relative to the C.L. and the boundary conditions at transverses' ends for y = 0 and y = *l* are identical. For such a symmetrically bending grillage, the calculation of longitudinals' elastic deflection curves is substantially simplified since it becomes possible to consider only one half of the grillage (relative to C.L.) which reduces the number of differential equations that determine the beams' elastic deflection curves.

For the even number of the longitudinals, the number of the corresponding equations is reduced twofold.

For the even number of longitudinals, the calculation of the coefficients of influence γ_{ij} can be done by using the formula for determination of a transverse's deflection resulting from the action of two symmetrically located forces at section y = c and y = *l* – c (index j) and find the deflection at section y (index i), see Figure 3.

For a freely supported transverse, the elastic deflection curve due to the effect of bending moments (caused by a lateral force P) is determined by the formula:

$$v(y) = \frac{Pl^3}{6Ei_0}\left[\begin{aligned} &\frac{y}{l}\left(3\frac{c(l-c)}{l^2} - \left(\frac{y}{l}\right)^2\right) + \\ &\Big\|_{\hat{c}}\left(\frac{y-c}{l}\right)^3 + \Big\|_{\widehat{l-c}}\left(\frac{y-(l-c)}{l}\right)^3 \end{aligned}\right] \quad (72)$$

For fixed transverse's ends, the elastic deflection curve due to the effect of bending moments is determined by the formula:

$$v(y) = \frac{Pl^3}{6Ei_0}\left[\begin{aligned} &\left(\frac{y}{l}\right)^2\left(3\frac{c(l-c)}{l^2} - \frac{y}{l}\right) + \\ &\Big\|_{\hat{c}}\left(\frac{y-c}{l}\right)^3 + \Big\|_{\widehat{l-c}}\left(\frac{y-(l-c)}{l}\right)^3 \end{aligned}\right] \quad (73)$$

The term after the first vertical lines should be used only for calculations of $v(y)$ in sections y > c. When the calculations are carried out for sections y > *l* – c, one should also use the term after the second vertical lines.

7. SOLUTION FOR ODD NUMBER OF LONGITUDINAL GIRDERS

The calculation of the buckling strength in this case is more complicated than for an even number of longitudinals. Since there is a longitudinal at C.L., one cannot use only half of the grillage in the calculations but the whole of it. This increases the calculations' complexity and time. However, with present computer technology this is not a problem.

The procedure for calculating the strength of a grillage is given in Appendix I.

8. CONCLUSION

The paper deals with strength of a grillage loaded in the following way: 1) by lateral load and in-plane compression load (in longitudinal direction); 2) only by lateral load, 3) only by in-plane compression. It consists of a system of prismatic girders crossing under 90^0. The compression load is taken by the longitudinal girders that are elastically fixed on rigid supports (transverse girders). A system of aggregated differential equations is derived for solution of the problems using the Lagrange method. It allows for replacement of the system of aggregated differential equations by a system of independent differential equations. These equations for the case of simultaneous action of lateral and longitudinal compression load (first problem) have the form of differential equations for bending of prismatic girders under lateral and longitudinal compression load laying on elastic foundation. When only lateral load exists (second problem), the form of these equations coincides with the form of differential equations for bending of girders laying on elastic foundation and loaded with lateral load alone. When only longitudinal compression load exists (third problem), the form of these equations coincides with the form of differential equations for buckling of girders laying on elastic foundation.

Solutions are given for bending of a grillage (the first two problems). Formulas are derived for calculation of the parameters of longitudinal girders' bending when girders' end sections are elastically fixed. Also, formulas are derived for calculation of the reaction forces at cross-points of transverse and longitudinal girders. When only longitudinal compression load exists (third problem), a solution is given for the connection between the coefficient of elastic foundation's rigidity and the Euler force.

The numerical examples given in the Appendices show very close results obtained by the proposed method and the FEM calculations. The maximal error is within 4 – 9% which allows for its application in engineering work, especially when FEM calculations are not available. It is worth noting that the proposed closed-form solution provides a conservative estimate relative to FEM.

Another potential application is in the early design stage of the ship's hull girder structure. One can run numerous calculations changing some of the geometric and material properties of the grillage under consideration and optimize its design.

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APPENDIX I

Calculation of the coefficients of influence γ_{ij}, coefficients of the form of main deflections ν_{im}, and coefficients of rigidity of the elastic foundation k_m

An example is given for a grillage with six equidistantly located longitudinal girders. The longitudinal girders are elastically fixed at rigid supports (e.g., transverse bulkheads) with the same coefficients of pliability. The transverse girders are freely supported at rigid supports.

Calculation of the coefficients γ_{ij} and β_i

Numbers 1, 2, and 3 are attached to longitudinals as shown in Figure 4 (symmetric girders have the same numbers.) To determine the coefficient γ_{ij} , one can use Eq. (72) in the following way:

$$\gamma_{ij} = \frac{1}{6}\left[\begin{array}{l}\frac{y}{l}\left(3\frac{c(l-c)}{l^2}-\left(\frac{y}{l}\right)^2\right)+ \\ \left\|_{\widehat{c}}\left(\frac{y-c}{l}\right)^3+\right\|_{\widehat{l-c}}\left(\frac{y-(l-c)}{l}\right)^3\end{array}\right] \tag{74}$$

In the example (see also Figure 4), Eq. (74) takes the form

$$\gamma_{ij} = \frac{y}{6l}\left(3\frac{c(l-c)}{l^2}-\left(\frac{y}{l}\right)^2\right) \tag{75}$$

By substituting in formula (75) the values of:
y = c = $3l$ / 7, one can find γ_{11} = 27/686
y = c = $2l$ / 7, one can find γ_{22} = 26/1029
y = c = $1l$ / 7, one can find γ_{33} = 17/2058
c = $3l$ / 7 and y = $2l$/7, one can find γ_{21} = 32/1029
c = $3l$ / 7 and y = $1l$/7, one can find γ_{31} = 5/294
c = $2l$ / 7 and y = $1l$/7, one can find γ_{32} = 29/2058
Note: $\gamma_{ij} = \gamma_{ji}$

When a beam is loaded with evenly distributed load, the deflection curve of freely supported beam due to the effect of bending moments will be

$$\nu(y) = \frac{Ql^3}{24Ei_0}\left[\frac{y}{l}-2\left(\frac{y}{l}\right)^3+\left(\frac{y}{l}\right)^4\right] \tag{76}$$

To determine the coefficient β_i one can use Eq. (76) in the following way:

$$\beta_i = \frac{1}{24}\left[\frac{y}{l}-2\left(\frac{y}{l}\right)^3+\left(\frac{y}{l}\right)^4\right] \tag{77}$$

By substituting in Eq.(77) the values of:
y = $3l$/7, one can find β_1 = 183/14406
y = $2l$/7, one can find β_2 = 295/28812
y = $1l$/7, one can find β_3 = 165/28812

Calculation of the coefficients of rigidity of the elastic foundation K_m

The determination of these coefficients requires finding the roots of the determinant of formula (16) in which the calculated coefficients γ_{ij} should be inputted. Thus, the determinant (16) will have the following form:

$$\begin{vmatrix}\frac{27}{686}-\lambda_m & \frac{32}{1029} & \frac{5}{294} \\ \frac{32}{1029} & \frac{26}{1029}-\lambda_m & \frac{29}{2058} \\ \frac{5}{294} & \frac{29}{2058} & \frac{17}{2058}-\lambda_m\end{vmatrix} = 0 \tag{78}$$

For this numerical example, formula (78) leads to the following cubic equation:

$$\lambda^3 - 7.2886.10^{-2}\lambda^2 + 7.3429.10^{-5}\lambda - 8.1456.10^{-9} = 0 \tag{79}$$

The three real roots of Eq. (79) are

$$\left.\begin{array}{ll}\lambda_1 = 7.1866E-02 & \lambda_2 = 8.9329E-04 \\ \lambda_3 = 1.2688E-04 & \end{array}\right\} \tag{80}$$

The existence of three real positive differing roots of λ means that the coefficients of rigidity k_m (determined by Eq. (13)) will have three real positive differing values.

Calculation of the main deflections' forms ν_{im}

Once the values of λ_1, λ_2, and λ_3 are calculated, the forms of the main deflections ν_{im} can be determined with accuracy up to any arbitrary co-multiplier based on the system of equations (15) which takes the following form for the grillage under consideration:

$$\left.\begin{array}{l}\left(\frac{27}{686}-\lambda_m\right)\nu_{1m}+\frac{32}{1029}\nu_{2m}+\frac{5}{294}\nu_{3m}=0 \\ \frac{32}{1029}\nu_{1m}+\left(\frac{26}{1029}-\lambda_m\right)\nu_{2m}+\frac{29}{2058}\nu_{3m}=0 \\ \frac{5}{294}\nu_{1m}+\frac{29}{2058}\nu_{2m}+\left(\frac{17}{2058}-\lambda_m\right)\nu_{3m}=0\end{array}\right\} \tag{81}$$

Sequentially substituting in this system of equations the values of λ_1, λ_2, and λ_3 and grant values to ν_{im} as ν_{11} = ν_{22} = ν_{33} = 1, one can find the values of the remaining forms of the main deflections. This is done in the following way:

- **when λ_1 = 7.1866E-02**

  ν_{11} is assumed as ν_{11} = 1. Then, one can use the second and third equation in formula (81) to obtain the following system of two equations:

$$\left.\begin{aligned} &\left(\frac{26}{1029}-\lambda_1\right)\nu_{21}+\frac{29}{2058}\nu_{31}=-\frac{32}{1029} \\ &\frac{29}{2058}\nu_{21}+\left(\frac{17}{2058}-\lambda_1\right)\nu_{31}=-\frac{5}{294} \end{aligned}\right\} \quad (82)$$

The solution of the system of two equations is $\nu_{21}=0.80194$; $\nu_{31}=0.44504$

- **when λ_2 = 8.9329E-04**

  ν_{22} is assumed as ν_{22} = 1. Then, one can use the first and second equation in formula (81) to obtain the following system of two equations:

$$\left.\begin{aligned} &\left(\frac{27}{686}-\lambda_3\right)\nu_{13}+\frac{32}{1029}\nu_{23}=-\frac{5}{294} \\ &\frac{32}{1029}\nu_{13}+\left(\frac{26}{1029}-\lambda_3\right)\nu_{23}=-\frac{29}{2058} \end{aligned}\right\} \quad (83)$$

  The solution of the system of two equations is ν_{12} = -1.80194; ν_{32} = 2.24698

- **when λ_3 = 1.2688E-04**

  ν_{33} is assumed as ν_{33} = 1. Then, one can use the first and third equation in formula (81) to obtain the following system of two equations:

$$\left.\begin{aligned} &\left(\frac{27}{686}-\lambda_2\right)\nu_{12}+\frac{5}{294}\nu_{32}=-\frac{32}{1029} \\ &\frac{5}{294}\nu_{12}+\left(\frac{17}{2058}-\lambda_2\right)\nu_{32}=-\frac{29}{2058} \end{aligned}\right\} \quad (84)$$

  The solution of the system of two equations is ν_{13} = 0.55496; ν_{23} = - 1.24698

The forms of main deflections should satisfy the condition of orthogonality (18) which can be rewritten in the following form:

$$\left.\begin{aligned} &\nu_{11}\nu_{12}+\nu_{21}\nu_{22}+\nu_{31}\nu_{32}=0 \quad (m=1,r=2) \\ &\nu_{11}\nu_{13}+\nu_{21}\nu_{23}+\nu_{31}\nu_{33}=0 \quad (m=1,r=3) \\ &\nu_{12}\nu_{13}+\nu_{22}\nu_{23}+\nu_{32}\nu_{33}=0 \quad (m=2,r=3) \end{aligned}\right\} \quad (85)$$

The accuracy of the calculations was checked by substituting ν_{ij} with the corresponding values given above. The maximal error in the calculations by formula (85) is 4.62E-06 (instead of zero) in the third equation in formula (85).

Once the forms of the main deflections are calculated, one can determine the intensity of the lateral pressure on longitudinal girders using formula (21). For the grillage under consideration, this formula has the following form:

$$q_m=\frac{Q}{a\,\lambda_m}\frac{\sum_{i=1}^{3}\beta_i\nu_{im}}{\sum_{i=1}^{3}\nu_{im}^2} \quad (86)$$

Taking into consideration formula (13), one can derive the following formula for q_m:

$$q_m=\frac{Q\,l^3\,k_m}{E\,i_0}\frac{\sum_{i=1}^{3}\beta_i\nu_{im}}{\sum_{i=1}^{3}\nu_{im}^2} \qquad Q=q\,a\,l \quad (87)$$

q = evenly distributed lateral load acting on the grillage.

The derived intensity of lateral load q_m and the coefficients of rigidity of the elastic foundation given above allow for finding the solution of the differential equation (9) that determines the main deflections $P_m(x)$. Further, using formula (7) for J_0 = J , one can find the deflection curve of each longitudinal girder.

APPENDIX II

Bending strength of a grillage loaded only with lateral load

The grillage in the previous example is taken into consideration. In such a case, the main deflections should satisfy the differential equation for bending of a beam on elastic foundation loaded by lateral load, i.e.,

$$E\,J_0\,P_m^{IV}(x)+k_m P_m(x)=q_m \quad (88)$$

The solution of the homogeneous Eq. (88) is searched in the following form:

$$\left.\begin{aligned} P_m(x)&=D_0V_0(\delta_m x)+D_1V_1(\delta_m x)+ \\ &+D_2V_2(\delta_m x)+D_3V_3(\delta_m x) \end{aligned}\right\} \quad (89)$$

D_i = constants

$V_i(\delta_m x)$ = linearly independent partial solutions of Eq. (88) for q_m = 0. These functions were determined by Puzirevsky N. P. [3] in the following way:

$$\left.\begin{aligned} V_0(\delta_m x)&=\mathrm{ch}(\delta_m x)\cos(\delta_m x) \\ V_1(\delta_m x)&=\frac{1}{\sqrt{2}}\left[\mathrm{ch}(\delta_m x)\sin(\delta_m x)+\mathrm{sh}(\delta_m x)\cos(\delta_m x)\right] \\ V_2(\delta_m x)&=\mathrm{sh}(\delta_m x)\sin(\delta_m x) \\ V_3(\delta_m x)&=\frac{1}{\sqrt{2}}\left[\mathrm{ch}(\delta_m x)\sin(\delta_m x)-\mathrm{sh}(\delta_m x)\cos(\delta_m x)\right] \end{aligned}\right\} \quad (90)$$

$$\delta_m = \sqrt[4]{\frac{k_m}{4EJ_0}} \tag{91}$$

The functions of Puzirevsky change over each other when subjected to differentiation relative to x in the following way:

$$\left.\begin{array}{ll} V_1^{/}(\delta x) = \sqrt{2}\,\delta V_0(\delta x) & V_2^{/}(\delta x) = \sqrt{2}\,\delta V_1(\delta x) \\ V_3^{/}(\delta x) = \sqrt{2}\,\delta V_2(\delta x) & V_0^{/}(\delta x) = -\sqrt{2}\,\delta V_3(\delta x) \end{array}\right\} \tag{92}$$

Then, using the function of Puzirevsky, the integral of Eq. (88) can be rewritten in the following form:

$$\left.\begin{array}{l} P_m(x) = D_0V_0(\delta x) + D_1V_1(\delta x) + \\ + D_2V_2(\delta x) + D_3V_3(\delta x) + q_m^o \end{array}\right\} \tag{93}$$

$$q_m^o = \frac{q_m}{k_m} = \frac{Ql^3}{Ei_0}\frac{\sum_{i=1}^{3}\beta_i\nu_{i,m}}{\sum_{i=1}^{3}\nu_{i,m}^2} = \frac{Ql^3}{Ei_0}\sigma_m \tag{94}$$

$$\sigma_m = \frac{\sum_{i=1}^{3}\beta_i\nu_{i,m}}{\sum_{i=1}^{3}\nu_{i,m}^2} \tag{95}$$

Let us determine the function $P_m(x)$ for a beam on rigid supports and elastically fixed with the same coefficients of pliability. Take the origin of the coordinate system in the middle of the beam's length. Since, in this case, the functions $P_m(x)$ will be symmetric relative to the origin of the coordinate system, the constants D_i for odd functions in Eq. (93) are equal to zero. Then,

$$P_m(x) = D_0V_0(\delta x) + D_2V_2(\delta x) + q_m^0 \tag{96}$$

The unknown constants D_0 and D_2 can be determined using the following boundary conditions:

$$\left.\begin{array}{l} \text{when } x = \dfrac{L}{2} \quad P_m\left(\dfrac{L}{2}\right) = 0 \\ P_m^{/}\left(\dfrac{L}{2}\right) = -\alpha\, EJ_0 P_m^{//}\left(\dfrac{L}{2}\right) \end{array}\right\} \tag{97}$$

On the basis of Eq. (50) and Eq. (51), one can present the boundary conditions in the following way:

$$P_m\left(\frac{L}{2}\right) = 0 \quad P_m^{/}\left(\frac{L}{2}\right) = -\frac{1-\zeta}{\zeta}\frac{L}{2}P_m^{//}\left(\frac{L}{2}\right) \tag{98}$$

On the basis of Eq. (92) and Eq. (96)

$$P_m^{/}(x) = \left[-D_0V_3(\delta_m x) + D_2V_1(\delta_m x)\right]\delta_m\sqrt{2} \tag{99}$$

$$P_m^{//}(x) = \left[-D_0V_2(\delta_m x) + D_2V_0(\delta_m x)\right]2\,\delta_m^2 \tag{100}$$

Also, taking into consideration the boundary conditions, one can obtain the following system of equations:

$$\left.\begin{array}{l} D_0V_0(u) + D_2V_2(u) = -q_m^0 \\ -D_0V_3(u) + D_2V_1(u) = \\ = -\dfrac{L}{2}\left(\dfrac{1-\zeta}{\zeta}\right)\left[-D_0V_2(u) + D_2V_0(u)\right]\delta_m\sqrt{2} \end{array}\right\} \tag{101}$$

Or

$$\left.\begin{array}{l} D_0V_0(u) + D_2V_2(u) = -q_m^0 \\ -D_0\left[\dfrac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_2(u) + V_3(u)\right] + \\ \quad + D_2\left[\dfrac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_0(u) + V_1(u)\right] = 0 \end{array}\right\} \tag{102}$$

$$u = \delta_m\frac{L}{2} \tag{103}$$

It follows that

$$D_2 = -q_m^0\frac{\frac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_2(u) + V_3(u)}{\Omega} \tag{104}$$

$$D_0 = -q_m^0\frac{\frac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_0(u) + V_1(u)}{\Omega} \tag{105}$$

$$\left.\begin{array}{l} \Omega = \dfrac{1-\zeta}{\zeta}\sqrt{2}\,u\left[V_0^2(u) + V_2^2(u)\right] + \\ \quad + V_1(u)V_0(u) + V_3(u)V_2(u) \end{array}\right\} \tag{106}$$

By substituting constants D_0 and D_2 in Equality (96), one can obtain:

$$\left.\begin{array}{l} P_m(x) = \\ = \dfrac{Ql^3}{Ei_0}\sigma_m\Big[1- \\ -\dfrac{C + V_1(u)V_0(\delta_m x) + V_3(u)V_2(\delta_m x)}{\Omega}\Big] \end{array}\right\} \tag{107}$$

$$\left.\begin{aligned}C=\frac{1-\zeta}{\zeta}\sqrt{2}\,u\big[V_0(u)V_0(\delta_m x)+\\+V_2(u)V_2(\delta_m x)\big]\end{aligned}\right\} \qquad (108)$$

The maximal value of the function $P_m(x)$ is for $x = 0$, i.e.,

$$P_m(0)=\frac{Ql^3}{Ei_0}\sigma_m\left[1-\varphi(u)\right] \qquad (109)$$

$$\varphi(u)=\frac{\frac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_0(u)+V_1(u)}{\Omega} \qquad (110)$$

The second derivative of the function $P_m(x)$ is

$$\left.\begin{aligned}&P_m^{//}(x)=\\&=\frac{Ql^3}{Ei_0}\sigma_m 2\delta^2\left[\frac{H+V_1(u)V_2(\delta_m x)}{\Omega}-\right.\\&\left.-\frac{V_3(u)V_0(\delta_m x)}{\Omega}\right]\end{aligned}\right\} \qquad (111)$$

$$\left.\begin{aligned}H=\frac{1-\zeta}{\zeta}\sqrt{2}\,u\big[V_0(u)V_2(\delta x)-\\-V_2(u)V_0(\delta x)\big]\end{aligned}\right\} \qquad (112)$$

Thus,

$$\left.\begin{aligned}&P_m^{//}(0)=\\&=-\frac{Ql^3}{Ei_0}\sigma_m 2\delta_m^2\frac{\frac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_2(u)+V_3(u)}{\Omega}\end{aligned}\right\} \qquad (113)$$

$$\left.\begin{aligned}&P_m^{//}\left(\frac{L}{2}\right)=\\&\frac{Ql^3}{Ei_0}\sigma_m 2\delta_m^2\frac{V_1(u)V_2(u)-V_3(u)V_2(u)}{\Omega}\end{aligned}\right\} \qquad (114)$$

Formulas (113) and (114) provide the basis for the following formulas:

$$P_m^{//}(0)EJ_0=\frac{8Ql^3J_0}{i_0L^2}u^2\sigma_m\psi_0(u) \qquad (115)$$

$$P_m^{//}\left(\frac{L}{2}\right)EJ_0=\frac{8Ql^3J_0}{i_0L^2}u^2\sigma_m\psi_1(u) \qquad (116)$$

Formulas (115) and (116) provide the basis for the following formulas:

$$\psi_0(u)=-\frac{\frac{1-\zeta}{\zeta}\sqrt{2}\,u\,V_2(u)+V_3(u)}{\Omega} \qquad (117)$$

$$\psi_1(u)=-\frac{V_1(u)V_2(u)-V_3(u)V_0(u)}{\Omega} \qquad (118)$$

Taking into consideration formula (7), when $J_0 = J$, the elements of bending of longitudinal girders would be determined by the following equalities:

Deflection at mid-length of the longitudinals

$$w_j(0)=\frac{Ql^3}{Ei_0}\sum_{m=1}^{n}\sigma_m\nu_{jm}\left[1-\varphi(u)\right] \qquad (119)$$

Bending moment at mid-length of the longitudinals

$$M_{x=0}=EJ_0w_j^{//}(0)=\frac{8Ql^3J_0}{L^2i_0}\sum_{m=1}^{n}u^2\sigma_m\nu_{jm}\psi_0(u) \qquad (120)$$

Bending moment at the ends of the longitudinals

$$M_{x=\frac{L}{2}}=EJ_0w_j^{//}\left(\frac{L}{2}\right)=\frac{8Ql^3J_0}{L^2i_0}\sum_{m=1}^{n}u^2\sigma_m\nu_{jm}\psi_1(u) \qquad (121)$$

To determine the reaction forces at cross-points of the girders in both directions, Rj(x), formula (5) is used where the function $w_j(x)$ is substituted using formula (7). Thus,

$$R_j(x)=a\sum_{m=1}^{3}\nu_{jm}\left[EJ_0P_m^{IV}(x)+TP_m^{II}(x)\right] \qquad (122)$$

Taking into consideration equality (9), one can obtain:

$$R_j(x)=a\sum_{m=1}^{3}\nu_{jm}\left[q_m-k_mP(x)\right] \qquad (123)$$

Then, by substituting $q_m(x)$ from formula (45) and using formula (109), one can obtain

$$R_j(0)=\frac{Ql^3}{Ei_0}a\sum_{m=1}^{3}\nu_{jm}k_m\sigma_m\varphi(u) \qquad (124)$$

The formulas for $\varphi(u)$, $\psi_0(u)$, and $\varphi_1(u)$ are given below for the case of freely supported girders ends ($\zeta = 0$) and for the case of fixed girders ends ($\zeta = 1$):

When $\zeta = 0$

$$\left.\begin{aligned} \varphi(u) &= \frac{V_0(u)}{[V_0(u)]^2 + [V_2(u)]^2} \\ \psi_0(u) &= \frac{V_2(u)}{[V_0(u)]^2 + [V_2(u)]^2} \\ \psi_1(u) &= 0 \end{aligned}\right\} \qquad (125)$$

When $\zeta = 1$

$$\left.\begin{aligned} \varphi(u) &= \frac{V_1(u)}{V_1(u)V_0(u) + V_3(u)V_2(u)} \\ \psi_0(u) &= \frac{V_3(u)}{V_1(u)V_0(u) + V_3(u)V_2(u)} \\ \psi_1(u) &= \frac{V_1(u)V_2(u) - V_3(u)V_0(u)}{V_1(u)V_0(u) + V_3(u)V_2(u)} \end{aligned}\right\} \qquad (126)$$

APPENDIX III

Bending of grillage under combined evenly distributed lateral and in-plane compression load

The calculations are related to integration of the differential equation (9). The formulas of functions $P_m(x)$ satisfying this equation in case of symmetric bending can be given in the form of Eq. (48) when $\eta_m^2 < 1$ or in the form of Eq. (58) when $\eta_m^2 > 1$. The components of bending for these two cases are:

When $\eta_m^2 < 1$

The coefficients $C_0^{/}$ and $C_2^{/}$ in Equality (48) become equal to

$$C_0^{/} = -q_m^o \frac{\nu_2 \sin\nu_2 + \frac{1-\zeta}{\zeta}\nu_2^2 \cos\nu_2}{\Delta_1} \qquad (127)$$

$$C_2^{/} = q_m^o \frac{\nu_1 \sin\nu_1 + \frac{1-\zeta}{\zeta}\nu_1^2 \cos\nu_1}{\Delta_1} \qquad (128)$$

$$\left.\begin{aligned} \Delta_1 &= \nu_2 \cos\nu_1 \sin\nu_2 - \nu_1 \cos\nu_2 \sin\nu_1 + \\ &+ \frac{1-\zeta}{\zeta}\left(\nu_2^2 - \nu_1^2\right)\cos\nu_2 \cos\nu_1 \end{aligned}\right\} \qquad (129)$$

Substituting the coefficients in Equality (48), one can obtain

$$\left.\begin{aligned} &P_m(x) = \\ &= q_m^0 \left[1 - \frac{\nu_2 \sin\nu_2 \cos(\alpha_1 x) - \nu_1 \sin\nu_1 \cos(\alpha_2 x)}{\Delta_1} + \right. \\ &\left. + \frac{\frac{1-\zeta}{\zeta}\left(\nu_2^2 \cos\nu_2 \cos(\alpha_1 x) - \nu_1^2 \cos\nu_1 \cos(\alpha_2 x)\right)}{\Delta_1}\right] \end{aligned}\right\} \qquad (130)$$

The second derivative of function $P_m(x)$ will be

$$\left.\begin{aligned} &P_m^{//}(x) = \\ &= -\frac{4}{L^2} q_m^0 \left[\frac{\nu_1 \nu_2^2 \sin\nu_1 \cos(\alpha_2 x) - \nu_2 \nu_1^2 \sin\nu_2 \cos(\alpha_1 x)}{\Delta_1} + \right. \\ &\left. + \frac{\frac{1-\zeta}{\zeta}\nu_1^2 \nu_2^2 \left(\cos\nu_1 \cos(\alpha_2 x) - \cos\nu_2 \cos(\alpha_1 x)\right)}{\Delta_1}\right] \end{aligned}\right\} \qquad (131)$$

The functions $P_m(x)$ for x = 0 and $P_m^{//}(x)$ for x = 0 and x = L/2 that are needed for calculation of the maximal deflection and bending moments of longitudinal girders will be:

$$\left.\begin{aligned} P_m(0) &= q_m^0 \left[1 - \varphi_0(\nu_1, \nu_2)\right] \\ P_m^{//}(0) &= -\frac{4}{L^2} q_m^0 \chi_0(\nu_1, \nu_2) \\ P_m^{//}\left(\frac{L}{2}\right) &= -\frac{4}{L^2} q_m^0 \chi_1(\nu_1, \nu_2) \end{aligned}\right\} \qquad (132)$$

where

$$\left.\begin{aligned} \varphi_0(\nu_1, \nu_2) &= \frac{\nu_2 \sin\nu_2 - \nu_1 \sin\nu_1}{\Delta_1} + \\ &+ \frac{\frac{1-\zeta}{\zeta}\left(\nu_2^2 \cos\nu_2 - \nu_1^2 \cos\nu_1\right)}{\Delta_1} \end{aligned}\right\} \qquad (133)$$

$$\left.\begin{aligned} \chi_0(\nu_1, \nu_2) &= \frac{\nu_1 \nu_2^2 \sin\nu_1 - \nu_2 \nu_1^2 \sin\nu_2}{\Delta_1} + \\ &+ \frac{\frac{1-\zeta}{\zeta}\nu_1^2 \nu_2^2 \left(\cos\nu_1 - \cos\nu_2\right)}{\Delta_1} \end{aligned}\right\} \qquad (134)$$

$$\chi_1(\nu_1, \nu_2) = \frac{\nu_1 \nu_2^2 \sin\nu_1 \cos\nu_2 - \nu_2 \nu_1^2 \sin\nu_2 \cos\nu_1}{\Delta_1} \qquad (135)$$

Thus, the above-given components of bending of longitudinal girders will be:

- Deflection of longitudinal girder at its mid-length (see formulas (7) and (46)

$$\left.\begin{aligned} & w_i(0) = \\ & \sum_{m=1}^{3} \nu_{im} P_m(0) = \sum_{m=1}^{3} \nu_{jm} q_m^0 \left[1-\varphi_0(\nu_1,\nu_2)\right] = \\ & = \frac{Ql^3}{E i_0} \sum_{m=1}^{3} \sigma_m \nu_{im} \left[1-\varphi_0(\nu_1,\nu_2)\right] \\ & \quad \text{where} \quad \sigma_m = \sum_{i=1}^{3} \beta_i \nu_{im} / \sum_{i=1}^{3} \nu_{im}^2 \end{aligned}\right\} \tag{136}$$

- Bending moment in "i[th]" girder

$$\left.\begin{aligned} & M_{x=0} = EJ_0 w_i^{//} = EJ_0 \sum_{m=1}^{3} \nu_{im} P_m^{//}(0) = \\ & = -EJ_0 \frac{4}{L^2} \sum_{m=1}^{3} \nu_{im} q_m^0 \chi_0(\nu_1,\nu_2) = \\ & = -\frac{4}{L^2} \frac{Ql^3 J_0}{i_0} \sum_{m=1}^{3} \nu_{im} \sigma_m \chi_0(\nu_1,\nu_2) \end{aligned}\right\} \tag{137}$$

- Bending moment at the fixed end of "i[th]" girder

$$\left.\begin{aligned} & M_{x=\frac{L}{2}} = EJ_0 \sum_{m=1}^{3} \nu_{im} P_m^{//}\left(\frac{L}{2}\right) = \\ & = -EJ_0 \frac{4}{L^2} \sum_{m=1}^{3} \nu_{im} q_m^0 \chi_1(\nu_1,\nu_2) = \\ & = -\frac{4}{L^2} \frac{Ql^3 J_0}{i_0} \sum_{m=1}^{3} \nu_{im} \sigma_m \chi_1(\nu_1,\nu_2) \end{aligned}\right\} \tag{138}$$

Based on Eqs. (5) and (7), one can find the reaction forces at crossing points of longitudinal and transverse girders when $J_0 = J$, i.e.,

$$a \sum_{m=1}^{3} \nu_{jm} \left[EJ_0 P_m^{IV}(x) + T P_m^{II}(x)\right] = R_j(x) \tag{139}$$

Taking into consideration Equality (9), the following formula for calculation of the reaction forces is derived:

$$R_j(x) = a \sum_{m=1}^{3} \nu_{jm} \left[q_m - k_m P_m(x)\right] \tag{140}$$

Substituting in Eq. (140) the formula for $P_m(0)$ given above, one can find the reaction force at the crossing point of longitudinal girders and the transverse girder located in the middle of the grillage.

$$R_j(0) = a \sum_{m=1}^{3} \nu_{jm} \left[q_m - k_m q_m^0 \left(1-\varphi_0(\nu_1,\nu_2)\right)\right] \tag{141}$$

Using Eq. (46), one can derive the following formula for $R_j(0)$:

$$\left.\begin{aligned} & R_j(0) = a \sum_{m=1}^{3} \nu_{jm} k_m q_m^0 \varphi_0(\nu_1,\nu_2) = \\ & = \frac{Ql^3}{E i_0} \sum_{m=1}^{3} \nu_{jm} k_m \sigma_m \varphi_0(\nu_1,\nu_2) \end{aligned}\right\} \tag{142}$$

The values of $\varphi_0(\nu_1,\nu_2)$, $\chi_0(\nu_1,\nu_2)$, and $\chi_1(\nu_1,\nu_2)$ for freely supported girder and fixed end girder are given below:

For freely supported girder:

$$\left.\begin{aligned} & \varphi_0(\nu_1,\nu_2) = \frac{\nu_2^2 \cos\nu_2 - \nu_1^2 \cos\nu_1}{\left(\nu_2^2-\nu_1^2\right)\cos\nu_1 \cos\nu_2} \\ & \chi_0(\nu_1,\nu_2) = \frac{\nu_1^2 \nu_2^2 \left(\cos\nu_1 - \cos\nu_2\right)}{\left(\nu_2^2-\nu_1^2\right)\cos\nu_1 \cos\nu_2} \\ & \chi_1(\nu_1,\nu_2) = 0 \end{aligned}\right\} \tag{143}$$

For girder with fixed end

$$\left.\begin{aligned} & \varphi_0(\nu_1,\nu_2) = \frac{\nu_2 \sin\nu_2 - \nu_1 \sin\nu_1}{\nu_2 \cos\nu_1 \sin\nu_2 - \nu_1 \cos\nu_2 \sin\nu_1} \\ & \chi_0(\nu_1,\nu_2) = \frac{\nu_1 \nu_2^2 \sin\nu_1 - \nu_2 \nu_1^2 \sin\nu_2}{\nu_2 \cos\nu_1 \sin\nu_2 - \nu_1 \cos\nu_2 \sin\nu_1} \\ & \chi_1(\nu_1,\nu_2) = \frac{\nu_1 \nu_2^2 \sin\nu_1 \cos\nu_2 - \nu_2 \nu_1^2 \sin\nu_2 \cos\nu_1}{\nu_2 \cos\nu_1 \sin\nu_2 - \nu_1 \cos\nu_2 \sin\nu_1} \end{aligned}\right\} \tag{144}$$

When $\eta_m^2 > 1$

The functions $P_m(x)$ for x = 0 and $P_m^{//}(x)$ for x = 0 and x = L/2 in Eqs. (58), (60), (67), and (68) become equal to

$$P_m(0) = q_m^0 \left[1 - \frac{1}{\Delta_2}\left(b_1 + \frac{1-\zeta}{\zeta} g\left(\frac{L}{2}\right)\right)\right] \tag{145}$$

$$\left.\begin{aligned} & P_m^{//}(0) = -\frac{4}{L^2 \Delta_2} q_m^0 \left\{b_1 f(0) - a_1 g(0) + \right. \\ & \left. + \frac{1-\zeta}{\zeta}\left[g\left(\frac{L}{2}\right) f(0) - f\left(\frac{L}{2}\right) g(0)\right]\right\} \end{aligned}\right\} \tag{146}$$

where a_1 and b_1 are determined by Eq. (65) and Δ_2 - by formula (69).

$$g(0) = 2\,\overline{\nu_1}\,\overline{\nu_2} \qquad f(0) = \overline{\nu_1}^2 - \overline{\nu_2}^2 \tag{147}$$

$$P_m^{//}\left(\frac{L}{2}\right) = -\frac{4}{L^2\Delta_2} q_m^0 \left[b_1 f\left(\frac{L}{2}\right) - a_1 g\left(\frac{L}{2}\right)\right] \qquad (148)$$

Substituting in Eqs. (146) and (15) the values of a_1, b_1, g(L/2), and f(L/2) (see Eqs. (65) and (66)), one can obtain:

$$\left.\begin{aligned} P_m^{//}(0) = &-\frac{4}{L^2\Delta_2} q_m^0 \Big\{ \left(\overline{v}_1^2 + \overline{v}_2^2\right)\left(\overline{v}_1 \operatorname{ch}\overline{v}_1 \sin\overline{v}_2 - \right. \\ &\left. - \overline{v}_2 \operatorname{sh}\overline{v}_1 \cos\overline{v}_2\right) + \\ &+ \frac{1-\zeta}{\zeta}\left[\left(\overline{v}_1^2 - \overline{v}_2^2\right) + \left(2\overline{v}_1\,\overline{v}_2\right)^2\right] \sin\overline{v}_1 \sin\overline{v}_2 \end{aligned}\right\} \qquad (149)$$

$$\left.\begin{aligned} P_m^{//}\left(\frac{L}{2}\right) = &-\frac{4}{L^2\Delta_2} q_m^0 \left[\left(\overline{v}_1^2 - \overline{v}_2^2\right)\left(\cos\overline{v}_2 \sin\overline{v}_2 + \right.\right. \\ &\left. + \operatorname{ch}\overline{v}_1 \operatorname{sh}\overline{v}_1\right) + \\ &\left. + 2\overline{v}_1\,\overline{v}_2\left(\overline{v}_2 \cos\overline{v}_2 \sin\overline{v}_2 - \overline{v}_1 \operatorname{sh}\overline{v}_1 \operatorname{ch}\overline{v}_1\right)\right] \end{aligned}\right\} \qquad (150)$$

The components of bending of longitudinal girders for $\eta_m^2 > 1$ are:

- Deflection of longitudinal girder at mid-length (see formulas (7) and (46)

$$\left.\begin{aligned} &w_i(0) = \\ &\sum_{m=1}^{3} v_{im} P_m(0) = \sum_{m=1}^{3} v_{im} q_m^0 \left[1 - \varphi_1\left(\overline{v}_1, \overline{v}_2\right)\right] = \\ &= \frac{Ql^3}{E i_0} \sum_{m=1}^{3} \sigma_m v_{im} \left[1 - \varphi_1\left(\overline{v}_1, \overline{v}_2\right)\right] \\ &\text{where} \quad \sigma_m = \frac{\sum_{j=1}^{3} \beta_i v_{im}}{\sum_{j=1}^{3} v_{im}^2} \end{aligned}\right\} \qquad (151)$$

- Bending Moment in "i^th^" girder

$$\left.\begin{aligned} M_{x=0} &= E J_0 w_i^{//}(0) = E J_0 \sum_{m=1}^{3} v_{im} P_m^{//}(0) = \\ &= -E J_0 \frac{4}{L^2} \sum_{m=1}^{3} v_{im} q_m^0 \theta_0\left(\overline{v}_1, \overline{v}_2\right) = \\ &= -\frac{4}{L^2} \frac{Q l^3 J_0}{i_0} \sum_{m=1}^{3} v_{im} \sigma_m \theta_0\left(\overline{v}_1, \overline{v}_2\right) \end{aligned}\right\} \qquad (152)$$

- Bending Moment at the fixed end of "i^th^" girder

$$\left.\begin{aligned} M_{x=\frac{L}{2}} &= E J_0 \sum_{m=1}^{3} v_{im} P_m^{//}\left(\frac{L}{2}\right) = \\ &= -E J_0 \frac{4}{L^2} \sum_{m=1}^{3} v_{im} q_m^0 \theta_1\left(\overline{v}_1, \overline{v}_2\right) = \\ &= -\frac{4}{L^2} \frac{Q l^3 J_0}{i_0} \sum_{m=1}^{3} v_{im} \sigma_m \theta_1\left(\overline{v}_1, \overline{v}_2\right) \end{aligned}\right\} \qquad (153)$$

- Reaction force

$$\left.\begin{aligned} R_i(0) &= \frac{Ql^3}{i_0} a \sum_{m=1}^{3} v_{im} k_m \sigma_m \varphi_1\left(\overline{v}_1, \overline{v}_2\right) \\ \varphi_1\left(\overline{v}_1, \overline{v}_2\right) &= \frac{1}{\Delta_2}\left[b_1 + \frac{1-\zeta}{\zeta} g\left(\frac{L}{2}\right)\right] \end{aligned}\right\} \qquad (154)$$

$$\left.\begin{aligned} \theta_0\left(\overline{v}_1, \overline{v}_2\right) = &\frac{1}{\Delta_2}\Big\{\left(\overline{v}_1^2 + \overline{v}_2^2\right)\left(\overline{v}_1 \operatorname{ch}\overline{v}_1 \sin\overline{v}_2 - \right. \\ &\left. - \overline{v}_2 \operatorname{sh}\overline{v}_1 \cos\overline{v}_2\right) + \frac{1-\zeta}{2}\Big[\left(\overline{v}_1^2 - \overline{v}_2^2\right) + \\ &+ \left(2\overline{v}_1\,\overline{v}_2\right)^2 \operatorname{sh}\overline{v}_1 \sin\overline{v}_2\Big]\Big\} \end{aligned}\right\} \qquad (155)$$

$$\left.\begin{aligned} \theta_1\left(\overline{v}_1, \overline{v}_2\right) = &\frac{1}{\Delta_2}\left[\left(\overline{v}_1^2 - \overline{v}_2^2\right)\left(\cos\overline{v}_2 \sin\overline{v}_2 + \right.\right. \\ &\left. + \operatorname{ch}\overline{v}_1 \operatorname{sh}\overline{v}_1\right) + \\ &+ 2\overline{v}_1\,\overline{v}_2\left(\overline{v}_2 \cos\overline{v}_2 \sin\overline{v}_2 - \overline{v}_1 \operatorname{sh}\overline{v}_1 \operatorname{ch}\overline{v}_1\right) \end{aligned}\right\} \qquad (156)$$

The values of $\varphi_1\left(\overline{v}_1, \overline{v}_2\right)$, $\theta_0\left(\overline{v}_1, \overline{v}_2\right)$, and $\theta_1\left(\overline{v}_1, \overline{v}_2\right)$ for simply supported girder and a girder with fixed ends are:

For simply supported girder:

$$\left.\begin{aligned} \varphi_1\left(\overline{v}_1, \overline{v}_2\right) &= \frac{g(L/2)}{2\overline{v}_1\,\overline{v}_2\left[\left(\operatorname{ch}\overline{v}_1 \cos\overline{v}_2\right)^2 + \left(\operatorname{sh}\overline{v}_1 \sin\overline{v}_2\right)^2\right]} \\ \theta_0\left(\overline{v}_1, \overline{v}_2\right) &= \frac{\left[\left(\overline{v}_1^2 - \overline{v}_2^2\right)^2 + \left(2\overline{v}_1\,\overline{v}_2\right)^2\right] \operatorname{sh}\overline{v}_1 \sin\overline{v}_2}{2\overline{v}_1\,\overline{v}_2\left[\left(\operatorname{ch}\overline{v}_1 \cos\overline{v}_2\right)^2 + \left(\operatorname{sh}\overline{v}_1 \sin\overline{v}_2\right)^2\right]} \\ \theta_1\left(\overline{v}_1, \overline{v}_2\right) &= 0 \end{aligned}\right\} \qquad (157)$$

For a girder with fixed ends:

$$\left.\begin{aligned}
&\varphi_1\left(\overline{\nu_1},\overline{\nu_2}\right)=\frac{b_1}{\overline{\nu_1}\sin\overline{\nu_2}\cos\overline{\nu_2}+\overline{\nu_2}\,\mathrm{ch}\overline{\nu_1}\mathrm{sh}\overline{\nu_1}}\\
&\theta_0\left(\overline{\nu_1},\overline{\nu_2}\right)=\\
&=\frac{\left(\overline{\nu_1}^2+\overline{\nu_2}^2\right)\left(\overline{\nu_1}\,\mathrm{ch}\,\overline{\nu_1}\sin\overline{\nu_2}-\overline{\nu_2}\,\mathrm{sh}\overline{\nu_1}\cos\overline{\nu_2}\right)}{\overline{\nu_1}\sin\overline{\nu_2}\cos\overline{\nu_2}+\overline{\nu_2}\,\mathrm{ch}\overline{\nu_1}\mathrm{sh}\overline{\nu_1}}\\
&\theta_1\left(\overline{\nu_1},\overline{\nu_2}\right)=\\
&=\frac{\left(\overline{\nu_1}^2-\overline{\nu_2}^2\right)\left(\cos\overline{\nu_2}\sin\overline{\nu_2}+\mathrm{ch}\overline{\nu_1}\mathrm{sh}\overline{\nu_1}\right)}{\overline{\nu_1}\sin\overline{\nu_2}\cos\overline{\nu_2}+\overline{\nu_2}\,\mathrm{ch}\overline{\nu_1}\mathrm{sh}\overline{\nu_1}}+\\
&+\frac{2\overline{\nu_1}\,\overline{\nu_2}\left(\overline{\nu_2}\cos\overline{\nu_2}\sin\overline{\nu_2}-\overline{\nu_1}\mathrm{ch}\overline{\nu_1}\mathrm{sh}\overline{\nu_1}\right)}{\overline{\nu_1}\sin\overline{\nu_2}\cos\overline{\nu_2}+\overline{\nu_2}\,\mathrm{ch}\overline{\nu_1}\mathrm{sh}\overline{\nu_1}}
\end{aligned}\right\}\quad(158)$$

The in-plane compression force T in the above-given formulas should be smaller than the Euler force T_E for the grillage under consideration. The Euler force (see Eq. (169)) for a beam lying on an elastic foundation can be determined for different boundary conditions by using data from Table 1 (see also Appendix IV).

When working with computers, it is more convenient to present Table 1 with equations. This can be done in the following way:

$$u=\frac{\theta+\gamma\mu+\varepsilon\mu^2}{1+\beta\mu+\chi\mu^2+\psi\mu^3}\quad(159)$$

$$\left.\begin{aligned}
&\theta=\left(\frac{1.520-0.528\zeta}{1-0.533\zeta}\right)^2\\
&\gamma=10^{-3}\begin{pmatrix}12.543-4.347\zeta^2+0.981\zeta^4+\\+27.654\zeta^6-27.534\zeta^8\end{pmatrix}\\
&\varepsilon=10^{-6}\begin{pmatrix}1.186-0.746\zeta^2+0.00507\zeta^4+\\+5.516\zeta^6-5.287\zeta^8\end{pmatrix}\\
&\beta=10^{-3}\begin{pmatrix}1.687-0.938\zeta^2+0.792\zeta^4+\\+3.410\zeta^6-3.875\zeta^8\end{pmatrix}\\
&\chi=10^{-8}\begin{pmatrix}6.785-4.590\zeta^2+0.191\zeta^4+\\+34.853\zeta^6-33.266\zeta^8\end{pmatrix}\\
&\psi=10^{-13}\begin{pmatrix}-1.132+0.852\zeta^2+0.208\zeta^4-\\-7.712\zeta^6+7.169\zeta^8\end{pmatrix}
\end{aligned}\right\}\quad(160)$$

The maximal error in calculating the parameter "u" by formulas (159) and (160) is between +4.0% and - 8.9%. These formulas provide the opportunity for calculating the parameter "u" for any value of μ and ζ with minor error which is an advantage relative to calculations using data in a table format.

APPENDIX IV

Buckling strength of a grillage under in-plane load alone

The derived formulas in the paper allow for calculating the buckling strength of a grillage for any boundary conditions for the longitudinal and transverse girders. The boundary conditions for transverse girders are taken into consideration when calculating the coefficients of influence γ_{ij} and β_i. Formulas (72) and (73) can be used to calculate the coefficients γ_{ij} for freely supported or fixed ends transverse girders loaded by concentrated forces. The coefficients β_i. can be calculated by the corresponding formulas determining the deflection of girders under uniformly distributed load. When the coefficients of influence γ_{ij} and β_i are calculated, one can determine the maximal value of the parameter λ_m (using the determinant (16)) and the corresponding coefficient of rigidity of the elastic foundation.

The determination of Euler force T_E is related to integrating the differential equation (9) for q(x) = 0:

$$E\,J_0P_m^{IV}(x)+T\,P_m^{II}(x)+\left(k_m\right)_{min}P_m(x)=0\quad(161)$$

For girders supported by absolutely rigid supports the condition $\eta^2<1$ is fulfilled. This allows for presentation of the function $P_m(x)$ in the following way:

$$\left.\begin{aligned}P_m(x)=C_0'\cos\left(\alpha_1 x\right)+C_1'\sin\left(\alpha_1 x\right)+\\+C_2'\cos\left(\alpha_2 x\right)+C_3'\sin\left(\alpha_2 x\right)\end{aligned}\right\}\quad(162)$$

where α_1 and α_2 are determined by Eq. (39).

When both ends of the girders have the same elastic fixity, the boundary conditions are:

$$\left.\begin{aligned}&\text{When } x=0\quad P_m(0)=0;\quad P_m^{/}(0)=\frac{1-\zeta}{\zeta}L\,P_m^{//}(0)\\&\text{When } x=L\quad P_m(L)=0;\quad P_m^{/}(L)=-\frac{1-\zeta}{\zeta}\frac{L}{2}P_m^{//}(L)\end{aligned}\right\}\quad(163)$$

The parameter ζ is related to the coefficient of pliability α as shown in Eq. (50).

By substituting the function $P_m(x)$ in conditions (163), one can obtain a system of linear homogeneous equations. The equality of its determinant to zero leads to a transcendent equation. Its minimal root determines the Euler force for the grillage.

A solution for simply supported girder (i.e., $\zeta = 0$) is given below:

When $\zeta = 0$, the system of homogeneous equations has the following form:

$$\left.\begin{aligned} C_1^{/} \sin(\alpha_1 L) + C_3^{/} \sin(\alpha_2 L) = 0 \\ C_1^{/} \alpha_1^2 \sin(\alpha_1 L) + C_3^{/} \alpha_2^2 \sin(\alpha_2 L) = 0 \end{aligned}\right\} \quad (164)$$

Hence,

$$\left(\alpha_2^2 - \alpha_1^2\right)\sin(\alpha_1 L)\sin(\alpha_2 L) = 0 \quad (165)$$

It follows that

$$\sin(\alpha_1 L) = 0 \quad ; \quad \alpha_1 L = j\,\pi \quad (166)$$

where j = integer

By substituting the value of α_1 in (166), one can obtain the following formula

$$T = \frac{\pi^2 E J_0}{L^2}\left(j^2 + \frac{\mu}{\pi^4 j^2}\right) \quad ; \quad \mu = \frac{(k_m)_{min} L^4}{E J_0} \quad (167)$$

The integer j that provides the minimal value of the Euler force T_E can be calculated from the inequality

$$j^2 (j-1)^2 \le \frac{\mu}{\pi^4} \le j^2 (j+1)^2 \quad (168)$$

When girders' ends are elastically fixed (i.e., $\zeta \neq 0$), the magnitude of the in-plane compression force T can be determined with data for the parameter "u" in Table 1. Once the parameter "u" is calculated, one can determine the Euler force T_E by the following equation:

$$T_E = \frac{2 u^2 E J_0}{L^2} \quad (169)$$

The Euler stress is

$$\sigma_E = \frac{T_E}{f} \quad \text{or} \quad \sigma_E = \frac{2 u^2 E J_0}{f L^2} \quad (170)$$

where f = longitudinal girder's cross-sectional area

Formula (170) can be used when the Euler stresses σ_E are smaller than the proportional limit. When σ_E is greater than the proportionality limit, one should take into consideration the effect of the deviation from the Hook's law in order to more accurately determine the stresses that cause buckling of the grillage. (The buckling stress calculated considering the effect of deviation from Hook's law is called critical buckling stress and is marked as σ_{cr}).

The effect of deviation from Hook's law is taken into consideration by introducing the coefficient φ by which the modulus of elasticity is multiplied. This coefficient can be calculated with the formulas given in Lokshin et al (2013). When the ratio $\eta_E = \sigma_E/\sigma_Y$ is known (σ_Y is material yield stress), the ratio $\eta_{cr} = \sigma_{cr}/\sigma_Y$ can be calculated by the formulas in Lokshin et al (2013) in the following way:

When $\sigma_Y = 2400$ kg/cm^2 = 23.54 KN/cm^2

$$\eta_{cr} = \frac{\sigma_{cr}}{\sigma_Y} = \frac{-0.044 + 1.437\dfrac{\sigma_E}{\sigma_Y}}{1 + 1.043\dfrac{\sigma_E}{\sigma_Y}} \quad (171)$$

When $\sigma_Y = 3000$ kg/cm^2 = 29.42 KN/cm^2

$$\eta_{cr} = \frac{\sigma_{cr}}{\sigma_Y} = \frac{-0.081 + 1.614\dfrac{\sigma_E}{\sigma_Y}}{1 + 0.945\dfrac{\sigma_E}{\sigma_Y}} \quad (172)$$

When $\sigma_Y = 4000$ kg/cm^2 = 39.23 KN/cm^2

$$\eta_{cr} = \frac{\sigma_{cr}}{\sigma_Y} = \frac{-0.059 + 1.474\dfrac{\sigma_E}{\sigma_Y}}{1 + 0.853\dfrac{\sigma_E}{\sigma_Y}} \quad (173)$$

As an example, the critical buckling stress is calculated for a grillage with the following dimensions:

Length L = 14 [m]
Width l = 3.5 [m]
Spacing of transverse girders a = 2.0 [m]
Number of transverse girders n = 6
Moment of inertia of transverse girders i_0 = 960 [cm^4]
Spacing of longitudinals b = 0.50 [m]
Moment of inertia of longitudinals J_0 = 600 [cm^4],
Cross-sectional area of longitudinals f = 35.5 [cm^2]
Longitudinals' ends are fixed at their supports [$\zeta = 1$]
Transverses are freely supported at their ends [$\zeta = 0$]
Modulus of elasticity E = 20600 [KN/cm^2]
Yield stress σ_Y = 29.42 [KN/cm^2]

- Determine the minimal value of the coefficient of rigidity of the elastic foundation k_m . It follows from Appendix I that the maximal value of the parameter λ_1 = 7.186E-02. Hence, the minimal value of the coefficient k_m can be calculated by Eq. (13)

$$k_{m,\,min} = \frac{E\, i_0}{a\, l^3 \lambda_1} = 23.06\text{E-04 [KN/cm}^2] \quad (174)$$

- Calculate $\mu = \dfrac{k_{m,\,min} L^4}{E J_0} = \dfrac{i_0 L^4}{J_0\, a\, l^3 \lambda_1} = 9974.95$ (see Eq.(167))
- From Table 1 for $\mu = 9974.95$ and $\zeta = 1$ one can find $u = 10.80$
- Calculate the Euler stress $\sigma_E = \dfrac{2u^2 E J_0}{f L^2} = 41.46$ [KN/cm^2]
- Calculate $\eta_E = \sigma_E / \sigma_Y = 1.409$
- Calculate $\eta_{cr} = 0.941$ (see Eq. (170))
- Calculate the coefficient $\varphi = \eta_{cr} / \eta_E = 0.6675$
- Calculate the critical buckling stress $\sigma_{cr} = 27.68$ [KN/cm^2]

Comparison between the calculated grillage deflections derived by the proposed method and FEM is given in Appendix V.

APPENDIX V

Comparison between the proposed theory and FEA simulations based on calculated grillage's deflection

The results presented here have been calculated for the grillage with the following dimensions:

Length L = 18.15 [m]
Width l = 17.85 [m]
Spacing of transverse girders a = 1.65 [m]
Number of transverse girders n = 10
Moment of inertia of transverse girders i_0 = 4,795,400 [cm^4]
Spacing of longitudinals b = 2.55 [m]
Number of longitudinals = 6
Moment of inertia of longitudinals J_0 = 7,787,349 [cm^4]
Transverses and longitudinals are freely supported at their ends.
Modulus of elasticity E = 20600 [kN/cm^2]
Lateral load q = 9 [t/m^2]
Longitudinal compressive load per one longitudinal T = 2.5 10^6 [N]

FEA simulations have been done using the software package SolidWorks. The FEA model is shown in Figure 5. Meshing was chosen based on convergence of results and was equal to 210,000. The results for deflection of the stringers Nos. 1, 2, and 3 are shown in Figure 6, Figure 7, and Figure 8, respectively, and also in Table 2. It can be seen that the difference between the proposed solution and FEA simulation is between 4% and 10%, with the FEA maximal deflections being smaller. The probable cause of the discrepancy is a 3-D effect of beam joints that is not taken into account in the proposed model. Agreement could be considered reasonable within engineering accuracy. Also, it is worth noting that the proposed closed-form solution provides a conservative estimate.

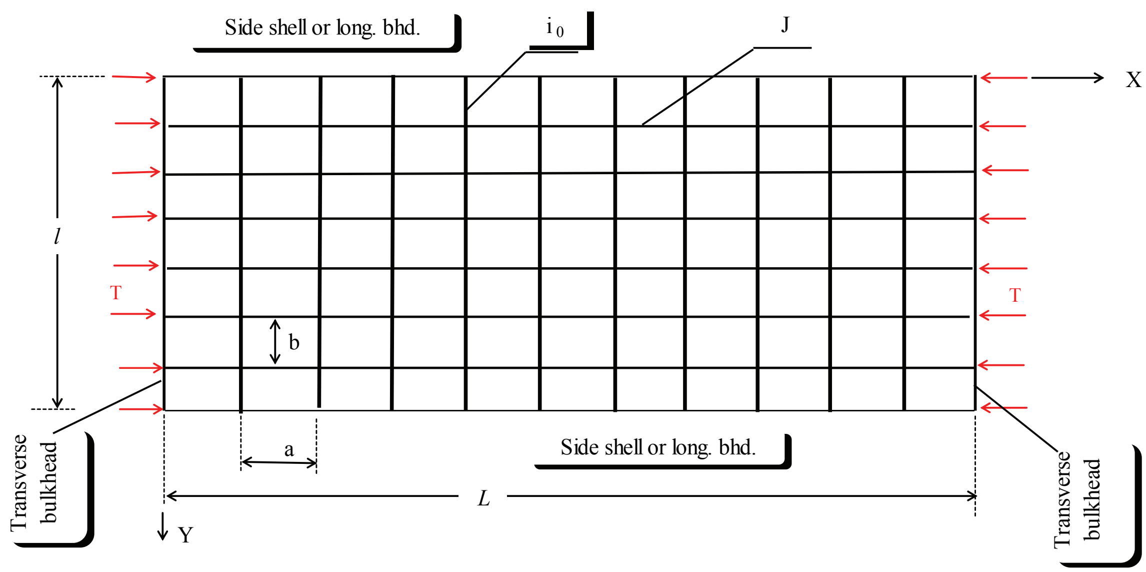

Figure 1 Components of a grillage (gross panel)

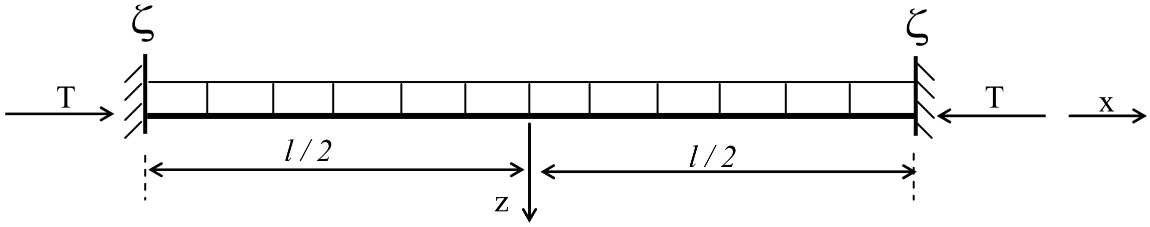

Figure 2 Coordinate system used in the calculations

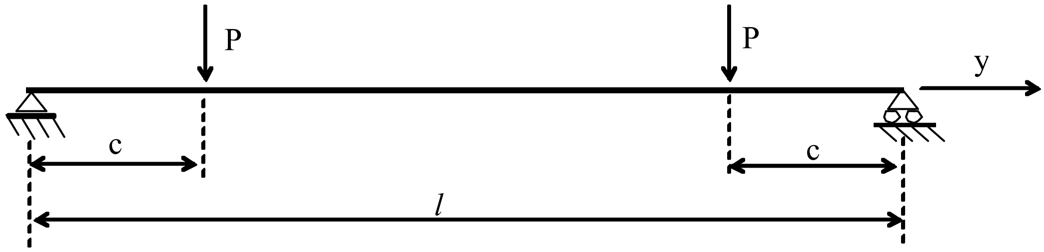

Figure 3 Drawing used in calculations of the coefficients of influence γ_{ij}

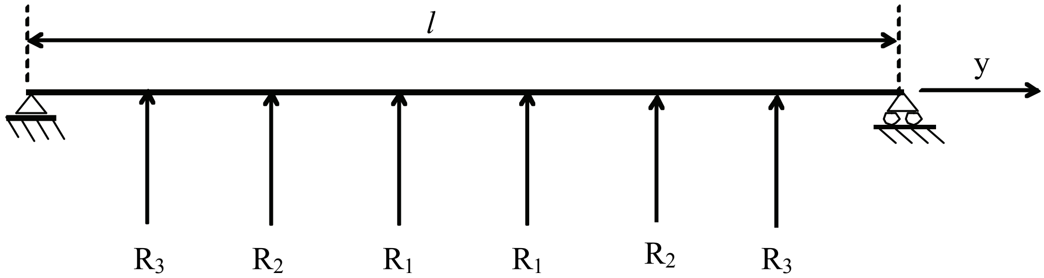

Figure 4 Numeration of longitudinal girders

Table 1 Values of the parameter u as a function of μ and ζ

μ	ζ = 0	ζ = 0.2	ζ = 0.4	ζ = 0.6	ζ = 0.8	ζ = 1.0
0.1	2.2226	2.4271	2.6980	3.0762	3.6358	4.4433
0.3	2.2249	2.4292	2.6998	3.0778	3.6371	4.4442
1	2.2328	2.4365	2.7064	3.0835	3.6416	4.4472
3	2.2554	2.4572	2.7249	3.0996	3.6546	4.4557
10	2.3327	2.5283	2.7890	3.1553	3.6997	4.4854
30	2.5406	2.7212	2.9644	3.3093	3.8253	4.5690
100	3.1624	3.3088	3.5096	3.7987	4.2335	4.8480
300	4.4870	4.5902	4.7311	4.9293	5.2077	5.5492
1000	5.6925	5.7761	5.8988	6.0944	6.4439	7.2591
3000	7.5983	7.6603	7.7484	7.8808	8.0876	8.3819
10000	10.0351	10.0826	10.1522	10.2629	10.4577	10.8117
30000	13.1888	13.2250	13.2784	13.3641	13.5180	13.8127
100000	17.8431	17.8700	17.9099	17.9752	18.0982	18.2884

Table 2 Deflections of longitudinal girders (stringers)

		Deflection [cm] at dimensionless abscissa $\bar{x}$					
stringer	Method used	$\bar{x}$ [-] = -1	$\bar{x}$ [-] = - 0.8	$\bar{x}$ [-] = - 0.6	$\bar{x}$ [-] = - 0.4	$\bar{x}$ [-] = - 0.2	$\bar{x}$ [-] = 0
1st stringer	Proposed theory	0	0.384	0.724	0.988	1.154	1.211
	FEA	0	0.396	0.704	0.924	1.056	1.100
2nd stringer	Proposed theory	0	0.312	0.588	0.800	0.934	0.979
	FEA	0	0.331	0.589	0.773	0.883	0.920
3rd stringer	Proposed theory	0	0.177	0.333	0.451	0.525	0.551
	FEA	0	0.191	0.339	0.445	0.509	0.530

Note: The dimensionless abscissa $\bar{x}$ is calculated as $\bar{x} = \frac{2x}{l}$ (see Figure 2)

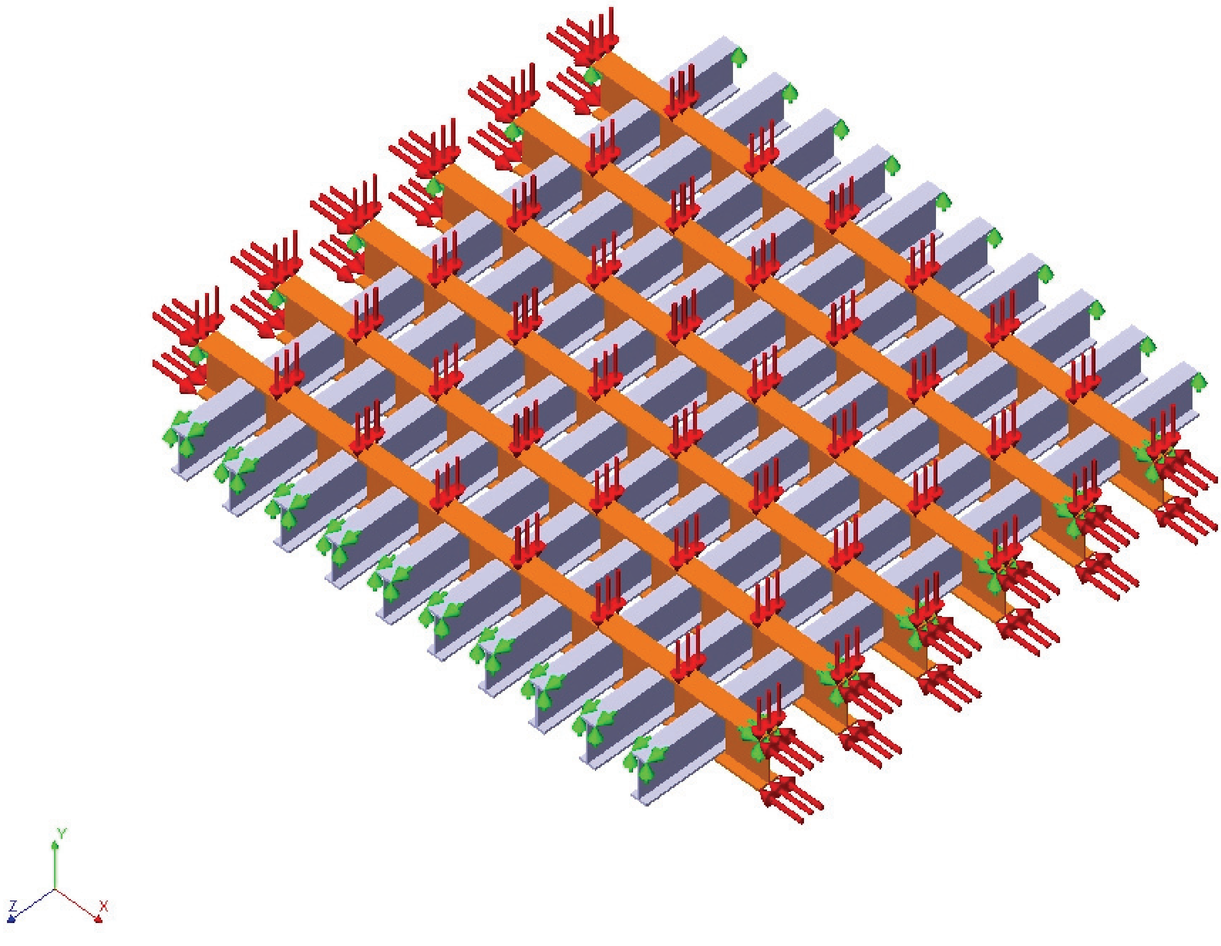

Figure 5 FEA model used in the calculations

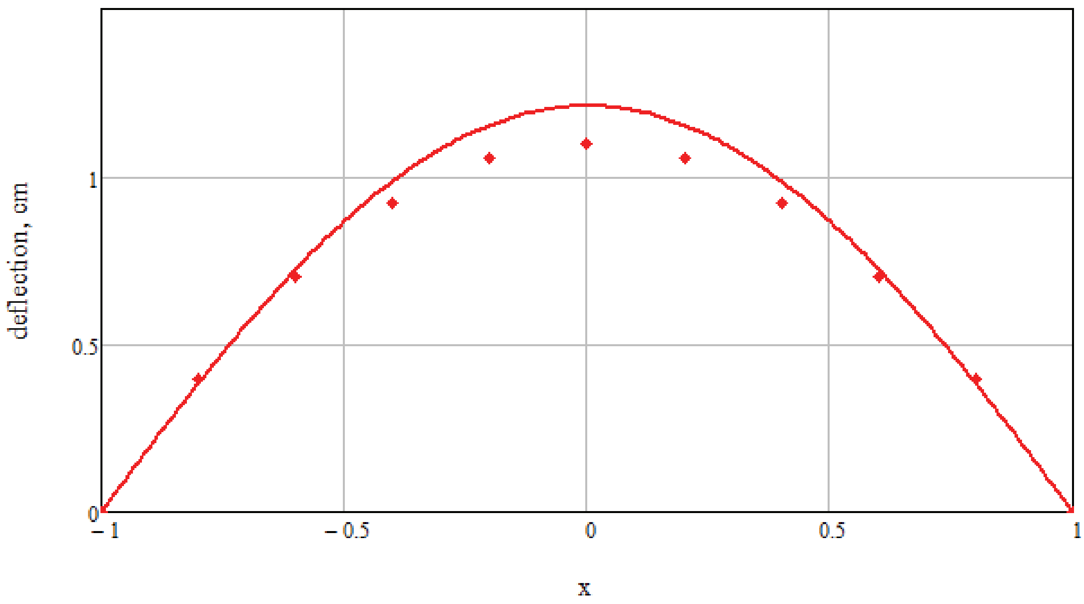

Figure 6 Deflection of the first stringer vs. coordinate x.
Solid line – proposed model; Solid dots – FEA.

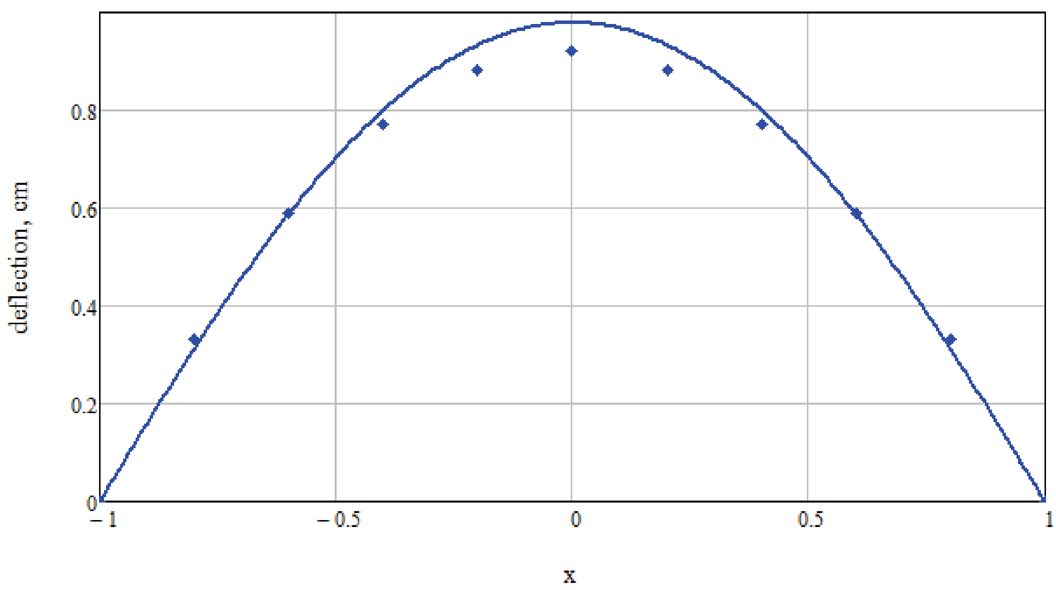

Figure 7 Deflection of the second stringer vs. coordinate x.
Solid line – proposed model; Solid dots – FEA.

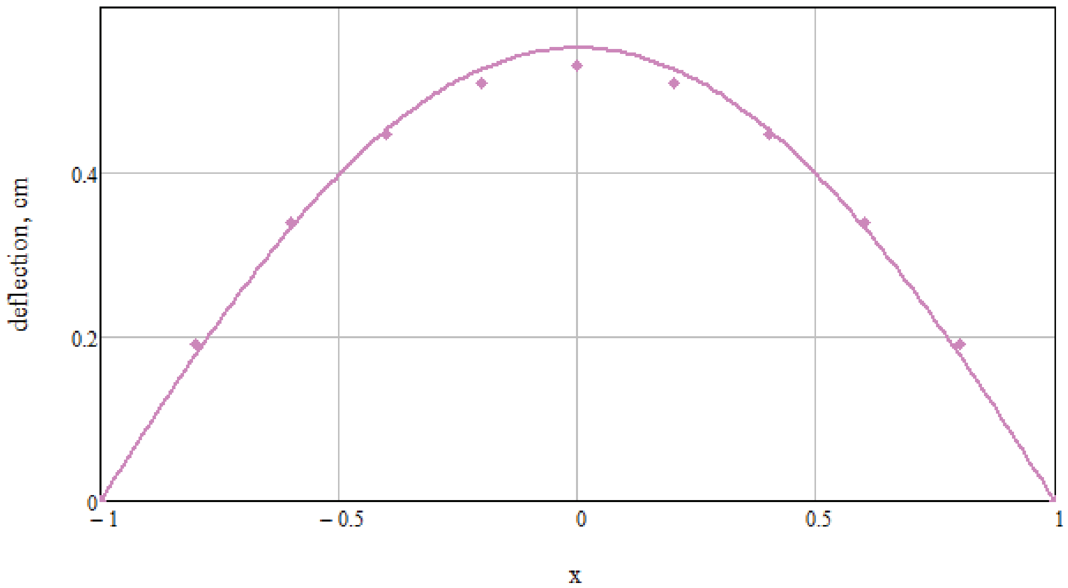

Figure 8 Deflection of the third stringer vs. coordinate x.
Solid line – proposed model; Solid dots – FEA.

