VISCOSITY IN SEAKEEPING

(DOI No: 10.3940/rina.ijme.2018.a2.473)

M Pawłowski, Gdansk School of Higher Education, Poland

SUMMARY

Application of strip theory for the prediction of ship motions in waves relies on sectional hydrodynamic coefficients; i.e. the added mass and damping coefficients. These coefficients apply to linearised problems and are normally computed for inviscid fluids. It is possible to account for viscosity but this cannot be done by the RANS equations, as in linear problems there is no room for turbulence. The hydrodynamic coefficients can include the effect of viscosity but this can be done rightly through the classic Navier–Stokes equations for laminar (non-turbulent) flows. For solving these equations commercial RANS software can be used, assuming no Reynolds stresses.

NOMENCLATURE

$\forall$	sectional area
A	amplitude of oscillations
a	sectional added mass
B	sectional breadth of body
b	sectional damping coefficient
c	sectional coefficient of stiffness
h	distance from body surface
$\boldsymbol{I}$	unit matrix
k	$= (\omega/2\nu)^{1/2}$ – wave number
k'	$= (\omega/\nu)^{1/2} = 2^{1/2}k$
K	$= k'\sqrt{i}$, constant
k	$= 2\pi/\lambda$ – wave number of regular wave
k	kinetic energy of turbulence
$\boldsymbol{k}$	unit vector of the z-axis
l	characteristic length of body
p	pressure
$\boldsymbol{P}$	stress tensor
$\overline{\boldsymbol{P}}$	stress tensor for smoothed motion
r	radius, distance from an axis
$\boldsymbol{R}$	tensor of turbulent fluctuations
Re	Reynolds number
RANS	Reynolds Averaged Navier–Stokes equations
$\boldsymbol{S}$	tensor of velocity
$\boldsymbol{S}_d$	tensor of deformations
T	period of oscillations
u	forward speed of ship
u_0	amplitude of velocity of oscillations
v	velocity field
β	incident angle between direction of wave propagation and ship speed
δ	$= 1/k = (2\nu/\omega)^{1/2}$ – skin depth (depth of penetration)
ϕ	velocity potential
λ	wave length
μ_t	dynamic coefficient of turbulent viscosity
μ	dynamic coefficient of viscosity
ν	$= \mu/\rho$ – kinematic coefficient of viscosity
ρ	density
ρ'	$k'r = 2^{1/2}kr$ – module of Kr
ρ	$= Kr$ – nondimensional radius (complex)
ω	$= 2\pi/T$ – circular frequency of oscillations
ω	circular frequency of regular wave

1. INTRODUCTION

A question arises if viscosity should be accounted for in ship's dynamics, particularly in seakeeping. When analysing a non-stationary ship's motion two hydrodynamic coefficients are used: the added mass and damping coefficients. Strictly speaking, they are applicable only to linearised equations of motion. In the case of inviscid fluids·, the field of velocity around the body has a potential ϕ, fulfilling the Laplace equation $\Delta\phi = 0$ along with boundary conditions. The said equation is linear, leaving no room for turbulence. In other words, such a velocity field is always smooth, both for viscous and inviscid fluids, clearly observed outside the boundary layer, where flow is potential, despite the viscosity of water. To show this, it is sufficient to observe that the Laplacian of velocity vanishes in potential flow, when $\boldsymbol{v} = \text{grad}\phi$. Namely:

$$\Delta\boldsymbol{v} = \Delta\,\text{grad}\,\phi = \text{grad}\,\Delta\phi = 0.$$

We could change here the sequence of differentiation and account for the fact that the potential fulfils the Laplace equation $\Delta\phi = 0$. Hence, outside the boundary layer the fluid behaves, as if it was inviscid. The same conclusion can be obtained resorting to a useful identity:

$$\Delta\boldsymbol{v} \equiv \text{grad}\,\text{div}\,\boldsymbol{v} - \text{rot}\,\text{rot}\,\boldsymbol{v}. \tag{1}$$

For incompressible ($\text{div}\,\boldsymbol{v} \equiv 0$) and irrotational ($\text{rot}\,\boldsymbol{v} \equiv 0$) flows, the Laplacian of velocity vanishes.

For rotational flows, the potential of velocity does not exist. Vorticity is due to viscosity, which comes to action in close vicinity of the wall, as e.g. in the boundary layer in a wall-bounded flow. In such a case, the governing equation for laminar flows, neglecting the unit mass force, is the Navier–Stokes equation:

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\tfrac{1}{\rho}\,\text{grad}\,p + \tfrac{1}{3}\nu\,\text{grad}\,\text{div}\,\boldsymbol{v} + \nu\Delta\boldsymbol{v}, \tag{2}$$

where μ and $\nu \equiv \mu/\rho$ are the dynamic and kinematic coefficients of viscosity, respectively, and ρ is the density of fluid. The two terms on the left hand-side represent the acceleration of a particle $d\boldsymbol{v}/dt$. This is a non-linear equation of motion. The only non-linear term – the second one on the left hand-side – is the acceleration related to the convection of the particle, known as the *convective acceleration*, which is the source of turbulence. For linearised equations, in which the non-linear term $(v \cdot \nabla)v$ is neglected, the averaging process introduces no Reynolds stresses and therefore the linearised equations of motion remain unchanged after averaging. In other words, solutions to linearised N–S equations, despite viscosity, are still smooth. Such equations are closed and need no turbulence models. For that reason, employing turbulence models for linear problems, which is widely applied in literature, is conceptually wrong, see for instance Salui *et al*. (2000), Quérard *et al*. (2009), and many others.

Linearised seakeeping analysis is normally based on the assumption of potential flow, applicable to inviscid fluids. Furthermore, in linear approach *physical* oscillations are replaced by *virtual* ones in which the body is assumed to be stationary, i.e. moving without oscillations. In such a case, motion of the fluid due to oscillations is depicted by the boundary condition on the surface of the body, crucial for the problem. That is, the normal component of fluid velocity on the outer surface of the body equals the normal component of velocity of the outer surface, which is such as on the body in real motion, completed by the boundary conditions on the free surface and in infinity. Fluid particles move (slide) along the surface of the body but this does not create any vorticity, as by assumption the fluid is inviscid.

Evidence shows (Salvesen *et al*, 1970) that the prediction of ship motions based on potential flow provides satisfactory results, except for roll, where viscous effects are considerable, particularly for damping. But, even then, there are serious doubts, if the hydrodynamic coefficients should be determined resorting to turbulence. Flow around a rolling ship contains vorticity, particularly for V-type and rectangular sections, but this does not mean necessarily that it is turbulent.

To overcome this problem a number of experimental and numerical methods have been developed for the prediction of roll damping. The most known is a semi-empirical method developed by Ikeda *et al*. (1978). The linearised-damping coefficient has been divided into a number of components, reflecting various effects. The idea of linearisation of the non-linear roll damping is thoroughly discussed by the author Pawłowski (2010). It is worth mentioning a substantial work on non-linear roll damping carried out by Spouge (1988). One common disadvantage of experimental methods is that the results obtained are limited to particular geometry of ships. To be free of this limitation, various numerical techniques are used for calculating the hydrodynamic coefficients for roll, such as discrete vortex method (DVM), random vortex method (RVM), and the Reynolds Averaged Navier–Stokes (RANS) equations, discussed briefly below. Vortex methods were popular in the past, as they require much smaller computing power than finite volume collocated grid approaches, part of which are RANS solvers. But nowadays, with rapid advancement in computing, FEM became widely applied, backed-up by the Volume of Fluid method, used for modelling the free surface. They are the best means for solving viscous flows. They can reproduce the creation of vorticity in the boundary layer and vortex shedding.

The hydrodynamic forces acting on the body in potential flows are found by integration of the dynamic pressure p, given by the Cauchy–Lagrange equation, over a wetted surface of the body in the mean position:

$$p = -\rho({}^{\partial\phi}/_{\partial t} + \tfrac{1}{2}v^2), \tag{3}$$

which results in the added mass m and damping coefficient N, both dependent on the circular frequency of *forced* oscillations $\omega = 2\pi/T$ [rad/s], where T is the period of oscillations, and $\boldsymbol{v} = \operatorname{grad}\phi$. The term $\tfrac{1}{2}v^2$ is usually neglected, as small quantity.

The convective acceleration can be expressed in two ways:

$$(v \cdot \nabla)v \equiv Sv \equiv \tfrac{1}{2}\operatorname{grad} v^2 - v \times \operatorname{rot} v, \tag{4}$$

where $S \equiv (\nabla \otimes v)^{\mathrm{T}}$ is the tensor of velocity. In potential flows $(v \cdot \nabla)v = \tfrac{1}{2}\operatorname{grad} v^2$.

Assuming that the convective acceleration is small relative to other terms, it can be neglected. The non-linear equation of motion (2) for viscous flows reduces then to the linear one:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho}\operatorname{grad} p + \frac{1}{3}\nu \operatorname{grad}\operatorname{div}\boldsymbol{v} + \nu\Delta\boldsymbol{v} \cdot \tag{5}$$

As such, it leaves no room for turbulence provided that the Reynolds number for a given case of flow is less than the critical value, i.e. when $\mathrm{Re} < \mathrm{Re}_{crit}$. Its solutions are smooth then, though not necessarily of laminar type, due to the complexity of flow induced by vorticity, in particular – by trailing vortices. Complex flows are not the same as turbulent. Nonetheless, the said equation is typically solved using various RANS solvers along with turbulence models, though a plain FEM would be entirely sufficient.

The right-hand side of equation (5) equals to ${}^{1}/_{\rho}\operatorname{Div}P$, where $P = -pI + 2\mu(S_d - \tfrac{1}{3}I\operatorname{div}v)$ is the stress tensor, p is the pressure, I is the unit matrix, and S_d is the tensor of deformation. The stress tensor P is essential for calculating the stress vector $p_n = Pn$, where n is a unit vector normal to given surface element.

Once, the instantaneous stress tensor P is obtained by integrating equation (5) along with the equation of continuity and boundary conditions, instantaneous values of the hydrodynamic forces and moments can be obtained by integrating the elementary forces $p_n dS$ over the wetted surface S of the submerged body.

Frequently, the pressure field p is either constant or a function of the velocity field $\boldsymbol{v}$. In such cases, equation (5) reduces to the *equation of diffusion*:

$$\partial \boldsymbol{v}/\partial t = \nu\Delta \boldsymbol{v}, \tag{6}$$

well known in theoretical physics.

2. AVERAGED NAVIER–STOKES EQUATIONS

As we can see, the linearised Navier–Stokes equation (5) is capable of providing realistic solutions for the hydrodynamic coefficients in roll, accounting for viscosity. But we have to be cautious here. The above statement is valid for laminar flows only, when $\mathrm{Re} < \mathrm{Re}_{crit}$, which is in practice the case. Otherwise, instability of flow (turbulence) happens due to the omitted convective acceleration $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$. The said instability opens room for turbulence stresses, important for analysis of the boundary layer, essential for the problem of steady resistance of the ship, and not for seakeeping.

When flow is turbulent the velocity field can be presented, as $\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}'$, where $\overline{\boldsymbol{v}}$ is the mean (time-averaged, *smoothed*) component, of laminar type, and $\boldsymbol{v}'$ is the *velocity of fluctuation* (turbulent pulsation), of stochastic nature. Similarly, the pressure $p = \overline{p} + p'$. Substituting $\boldsymbol{v}$ and p to equation (2), after averaging the following is obtained for the equation of turbulent flow:

$$\frac{\partial \overline{\boldsymbol{v}}}{\partial t} + \overline{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}} = -\tfrac{1}{\rho}\,\mathrm{grad}\,\overline{p} + \tfrac{1}{3}\nu\,\mathrm{grad}\,\mathrm{div}\,\overline{\boldsymbol{v}} + \nu\Delta\overline{\boldsymbol{v}}\cdot \tag{7}$$

If $\mathrm{div}\,\boldsymbol{v}' = 0$, the averaged convective acceleration equals:

$$\begin{aligned}\overline{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}} &= (\overline{\boldsymbol{v}}\cdot\nabla)\overline{\boldsymbol{v}} + \overline{(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}'} = \\ &= (\overline{\boldsymbol{v}}\cdot\nabla)\overline{\boldsymbol{v}} + \nabla\left(\overline{\boldsymbol{v}'\otimes\boldsymbol{v}'}\right) = (\overline{\boldsymbol{v}}\cdot\nabla)\overline{\boldsymbol{v}} + \mathrm{Div}\,\boldsymbol{R},\end{aligned} \tag{8}$$

where $R \equiv \overline{(v'\otimes v')}$ is a dyad (tensor) of averaged turbulent fluctuations made up of two the very same vectors $\boldsymbol{v}'$ and $\boldsymbol{v}'$, whose elements $R_{ij} = \langle v'_i v'_j \rangle$ are averaged with respect to time:

$$\boldsymbol{R} = \begin{bmatrix} \overline{v_1'^2} & \overline{v_1'v_2'} & \overline{v_1'v_3'} \\ \overline{v_2'v_1'} & \overline{v_2'^2} & \overline{v_2'v_3'} \\ \overline{v_3'v_1'} & \overline{v_3'v_2'} & \overline{v_3'^2} \end{bmatrix} = \begin{bmatrix} \overline{v_p'^2} & 0 & 0 \\ 0 & \overline{v_q'^2} & 0 \\ 0 & 0 & \overline{v_r'^2} \end{bmatrix},$$

where v'_1, v'_2, v'_3 are components of the velocity of fluctuation at any orthogonal co-ordinate system, while v'_p, v'_q, v'_r are components of fluctuations in the principal system. For a stationary flow (in terms of smoothed quantities) this tensor is independent of time, but dependent on space point, i.e. $R_{ij} = R_{ij}(\boldsymbol{r})$. Further, R is a symmetric tensor for which the principal co-ordinate system pqr can be found in which the cross-product elements vanish. For this reason, the tensor of turbulent fluctuations has *three* degrees of freedom (not six but three), i.e. three its elements are independent of the remaining ones. In other words, to define the tensor of fluctuations R it is sufficient to define the three degrees of freedom, e.g. the three principal values on the main diagonal. This opens room for modelling turbulence.

The easiest case for modelling is isotropic turbulence, invariant under rotations, with equal principal values on the main diagonal, as only one quantity has to be estimated. However, isotropic turbulence does not occur in technical applications. Another easy case for modelling is flow through a pipeline, where only one element of tensor R is meaningful: $R_{13} \equiv R_{rz}$.

The tensor of fluctuations R has the same properties, as any symmetric tensor of the third order, as e.g. the stress tensor known from strength of materials. Hence, the sum of elements on the main diagonal $R_{11} + R_{22} + R_{33}$ is independent of the orientation of the system and equals $2k$, where k is the turbulence kinetic energy (TKE). This quantity value is one of the three invariants of tensor R and the most important characteristic of turbulence. TKE vanishes on the surface of the body and on the outer surface of the boundary layer. Regarding the principal axes, it can be assumed they are parallel to the principal axes of the tensor of deformations S_d for smoothed velocity field v^{-}.

Elements on the main diagonal R_{11}, R_{22}, R_{33} have the meaning of variance of fluctuations, whereas R_{12}, R_{13}, R_{23} have the meaning of covariance, representing correlation between fluctuating velocities.

For isotropic turbulence, elements on the main diagonal $R_{11} = R_{22} = R_{33} = ⅔k$ are the same, the cross product terms $R_{12} = R_{13} = R_{23} = 0$ vanish, and the principal axes are indefinite. The fluid has then no turbulence sheer stresses. For instance, if flow through a pipeline featured isotropic turbulence, the velocity profile would be the same, as for laminar flow, which is contradictory to reality.

Substituting formulation (8) to equation (7), the following is obtained for the equation of motion for a turbulent flow:

$$\begin{aligned}&\frac{\partial \overline{\boldsymbol{v}}}{\partial t} + (\overline{\boldsymbol{v}}\cdot\nabla)\overline{\boldsymbol{v}} = \\ &-\tfrac{1}{\rho}\,\mathrm{grad}\,\overline{p} + \tfrac{1}{3}\nu\,\mathrm{grad}\,\mathrm{div}\,\overline{\boldsymbol{v}} + \nu\Delta\overline{\boldsymbol{v}} - \mathrm{Div}\boldsymbol{R},\end{aligned} \tag{9}$$

known as the RANS equations, where a bar above the notation denotes a smoothed (averaged) quantity. The first three terms on the right-hand side equal to $^1/_\rho \mathrm{Div}\overline{P}$, where $\overline{P} = -\overline{p}I + 2\mu(S_d - ⅓I\operatorname{div}\overline{v})$ is the stress tensor for smoothed motion, $\overline{p}$ and S_d is the pressure and tensor of deformation for smoothed velocity field, respectively. With this notation, equation (9) can be written as:

$$\frac{d\overline{\boldsymbol{v}}}{dt} = \frac{1}{\rho}\operatorname{Div}\left(\overline{\boldsymbol{P}} - \rho\boldsymbol{R}\right), \tag{10}$$

where $\overline{P} - \rho R = -(\overline{p} + ⅔\mu\operatorname{div}\overline{v})I + 2\mu S_d - \rho R$, and

$$-\rho\boldsymbol{R} = \begin{bmatrix} -\rho\overline{v_1'v_1'} & -\rho\overline{v_1'v_2'} & -\rho\overline{v_1'v_3'} \\ -\rho\overline{v_2'v_1'} & -\rho\overline{v_2'v_2'} & -\rho\overline{v_2'v_3'} \\ -\rho\overline{v_3'v_1'} & -\rho\overline{v_3'v_2'} & -\rho\overline{v_3'v_3'} \end{bmatrix}. \tag{11}$$

The tensor $-\rho R$ is the additional apparent stress tensor, owing to the fluctuating velocity field, called the *turbulent stress tensor*, or the *tensor of Reynolds stresses*.

The pressure terms on the main diagonal are negative, while the tangential terms must have the *same* sign, as correspon-ding elements of the tensor of deformation for smoothed velocity field $\boldsymbol{S_d}$, which follows from their physical meaning. By the very nature of things, turbulence increases the absolute value of all the stresses.

The divergence of the tensor of fluctuations R is directly connected to vorticity, hidden in the RANS equations. This is clearly seen if one resorts to equation (4):

$$\operatorname{Div}R = \overline{(v'\cdot\nabla)v'} \equiv ½\operatorname{grad}\langle v'^2\rangle - \langle v'\times\operatorname{rot}v'\rangle, \tag{12}$$

where $\langle v'^2\rangle = R_{11} + R_{22} + R_{33} = 2k$. Equation (12) shows the importance of vorticity in generation of the Reynolds stresses. These stresses require relating them to characteristics of the averaged velocity field to close the RANS equation for solving, which has led to the creation of many turbulence models.

There are generally two types of turbulence models: algebraic and Reynolds-stress models. Models of the first group (k–ε, k–ω) are very popular. They resort to the idea of *turbulent viscosity*, introduced by Boussinesq in 1877, shortly discussed below.

The Reynolds stress tensor can be presented as:

$$-\rho R = -p_t I + (-\rho R + p_t I),$$

where $p_t = ⅔\rho k$ is the mean turbulent pressure. It is assumed that the tensor in parentheses is proportional to the net tensor of deformation in smoothed flow, i.e.:

$$(-\rho R + p_t I) = 2\mu_t(S_d - ⅓I\operatorname{div}\overline{v}), \tag{13}$$

where μ_t is the so-called (dynamic coefficient of) *turbulent viscosity*, termed also the *eddy viscosity*. Equation (13) is strict, if the principal directions of the two tensors are the same, which is true in the case of straight-linear flows. The whole stress tensor takes then the form:

$$\begin{aligned}\overline{P} - \rho R = -\left[\overline{p} + p_t + ⅔(\mu + \mu_t)\operatorname{div}\overline{v}\right]I + \\ + (\mu + \mu_t)2S_d.\end{aligned} \tag{14}$$

Velocity field is defined by equation (10) in which the divergence of the whole stress tensor is given by the following formulation:

$$\begin{aligned}\operatorname{Div}(P - \rho R) = -\operatorname{grad}(\overline{p} + p_t) + (\mu + \mu_t)\Delta\overline{v} + \\ + ⅓(\mu + \mu_t)\operatorname{grad}\operatorname{div}\overline{v} - ⅔\operatorname{div}\overline{v}\operatorname{grad}\mu_t + 2\operatorname{grad}\mu_t S_d.\end{aligned} \tag{15}$$

The above equation requires some comments. Firstly, the Reynolds stress tensor $-\rho R$ and the tensor $(-\rho R + p_t I)$ yield the same sheer stresses. Secondly, in close vicinity of the surface of the body the turbulent pressure $p_t = ⅔\rho k$ varies mainly in a plane normal to the average velocity $\overline{\boldsymbol{v}}$. Therefore, for the sake of simplicity the longitudinal component of gradp_t (along $\overline{\boldsymbol{v}}$) can be neglected. A good example is flow through a boundary layer or a pipeline, where the longitudinal component of the gradp_t is ignored. The rejection of p_t does not affect the sheer stresses.

TKE and turbulent viscosity vanish on surface of the body and outside the boundary layer. Equation (15) implies that a turbulent flow can be viewed as the laminar one with a varying viscosity. The latter depends on the Reynolds number and distance from the surface of the body, with a maximum value somewhere inside the boundary layer. However, none of the publications available in literature, including the two mentioned earlier, shows how turbulent viscosity varies across the boundary layer or how its thickness varies along the body.

In the k–ε model the kinematic coefficient of turbulent viscosity $\nu_t \equiv \mu_t/\rho$ is approximated by $\nu_t = 0.09k^2/\varepsilon$, where k is the TKE and ε is the rate of dissipation of TKE. The model comprises as well two PDE (transport equations) for k and ε. The equation for ν_t is clearly ill conditioned on extremes of the boundary layer, as k and ε vanish. Therefore k–ε models have generally poor performance in realistic flow situations, discussed in detail by McDonough (2007). Somewhat better results provide Reynolds stress models in which transport equations are derived for elements R_{ij}, but they are expensive in terms of computational time. Still, a complete description of turbulence remains one of the unsolved problems in classical physics.

Turbulence models have been developed mainly for stationary flows within the boundary layer. Stationary – in terms of smoothed quantities. There are doubts if turbulence exists at all outside the body (in the wake), behind the separation point. These doubts are due to decay of the normal derivative of velocity at the separation point and, what goes with it, the vanishing of turbulent stresses just at this point. This follows immediately from Prandtl's mixing-length hypothesis.

No turbulence models exist for non-stationary flows, and oscillatory motions in particular. Further, these models would have to be time dependent, which is not feasible. Presumably, all models have been developed and calibrated for stationary flows. Use of any turbulence model for oscillating bodies is therefore strongly speculative, and of little real merit.

3. HYDRODYNAMIC COEFFICIENTS

The idea of the hydrodynamic coefficients, i.e. the added mass and damping coefficients is solely applicable to linear problems, in which the body hardly moves, if at all. In such circumstances there is no room for developing turbulence. Hence, it can be assumed that there are no Reynolds stresses at all, which reduces RANS equation (9) to regular N–S equations (2). Hence, the same solver can be used for solving both the N–S and RANS equations, e.g. commercial RANS solvers ANSYSCFX10.0, COMET, CFDSHIP-IOWA, etc., assuming no Reynolds stresses.

In non-linear harmonic oscillations of finite amplitude in calm water the hydrodynamic coefficients are not constant in respect to time and, apart from that, they are dependent on the amplitude of oscillations. Consequently, they have to be averaged over time.

In linear problems the body is stationary, performing no oscillations. Its motion is depicted by the kinematic boundary conditions. Though it is acceptable to assume that the body physically oscillates calculations become cumbersome and results less accurate.

In the case the body physically oscillates, equations for the hydrodynamic sectional forces are as follows:

$$\begin{aligned}
&\text{sway:} && -a_{22}\ddot{y}-b_{22}\dot{y}=F_y(t),\\
&\text{heave:} && -a_{33}\ddot{z}-b_{33}\dot{z}-c_{33}z=F_z(t), \qquad (16)\\
&\text{roll:} && -a_{44}\ddot{\alpha}-b_{44}\dot{\alpha}-c_{44}\alpha=M_x(t),
\end{aligned}$$

where in general a is the added mass (in kg/m – for a_{22} and a_{33}, in kgm – for a_{44}), b is the damping coefficient (in kg/sm – for N_{22} and N_{33}, and in kgm/s – for N_{44}), and c is the coefficient of stiffness, all per unit length. For heave, $c_{33} = B\rho g$ [N/m^2], where B is breadth of the body at the waterline. For roll, $c_{44} = \rho g\forall GM$ [N], where $\forall$ is the sectional underwater area, and GM is the height of the metacentre above the waterline (the origin G is normally taken at the centreline of the waterline) .The right hand-sides represent the hydrodynamic forces, obtained by measurements or by integration of the pressure p and tangential stresses τ over the wetted surface of the body.

In the case of virtual oscillations, the hydrostatic terms vanish, as $z = \alpha = 0$. The hydrodynamic forces are obtained by integrating the linearized pressure over the wetted surface of the body, the same as in the *mean* position of the oscillating body (Faltinsen, 1990; Salvesen *et al*., 1970).

The main difficulty is to extract from the whole dynamic force, the time dependent harmonic part, which should be resolved next into the inertial and damping components.

The hydrodynamic forces on the right hand sides of equations (16), i.e. $F_y(t)$, $F_z(t)$, $M_x(t)$ are provided measurements or by numerical calculations per unit length as time histories. They are calculated for forced harmonic oscillations in calm water for y, z, and α, of general form $A\sin\omega t$, with given amplitude A and circular frequency ω.

Applying Fourier analysis to equations (16), after performing simple mathematics, we get in general the following expressions for the sectional added mass and damping coefficients:

$$a=\frac{c}{\omega^2}+\frac{1}{A\pi\omega}\int_t^{t+T}F(t)\sin\omega t\,dt, \qquad (17)$$

$$b=\frac{1}{A\pi}\int_t^{t+T}F(t)\cos\omega t\,dt, \qquad (18)$$

where $T = 2\pi/\omega$ is the period of oscillations, A is the amplitude of forced oscillations, and $F(t)$ stands for the time varying hydrodynamic force or moment for given circular frequency ω and amplitude A. These forces deviate from harmonic runs, if the equations of motion are non-linear. The coefficient of stiffness c in equation (17) is treated as known quantity.

4. THE EFFECT OF FORWARD SPEED

Regarding seakeeping, most research has been devoted to calculating the hydrodynamic coefficients for a stationary ship oscillating in calm water. As will be shown later, in such a case the boundary layer is marginal, and can be ignored. The hydrodynamic coefficients, except for roll, can be therefore calculated traditionally, assuming no viscosity of water, and, what goes with it, assuming potential flow, governed by the Laplace equation $\Delta\phi = 0$. For roll, however, the hydrodynamic coefficients have to be calculated accounting for viscosity.

Measured values of the added mass a_{44} are *smaller*, while for the damping coefficient b_{44} – *higher* from those for potential flow in proportion to the amplitude of roll (Figure. 1). The effect is noticeable. Measured values, denoted by points for three different amplitudes A: 0.05, 0.1, 0.2 rad (2.875°, 5,75°, 11.5°) are taken from Vugts (1968). Values for potential flow (thick line) were obtained by Dudziak (1988) with the help of multi-pole potentials. To obtain the hydrodynamic coefficients with the effect of viscosity, a FEM can be used for the integration of equations of conservation, or any RANS solver, ignoring the Reynolds stresses.

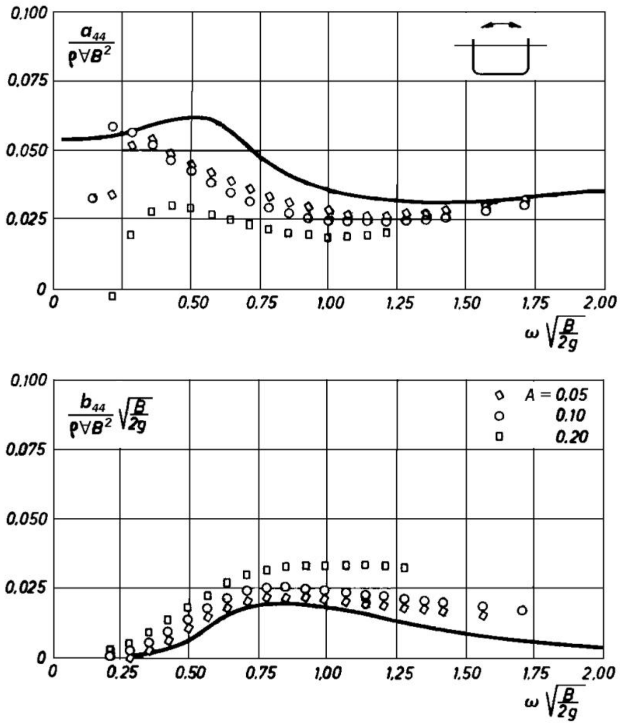

Figure. 1. Nondimensional hydrodynamic coefficients for roll of a rectangular section

For a ship advancing with forward speed in regular seas, the hydrodynamic coefficients can be calculated in two ways. 1), assuming that they are the same as for forced oscillations in calm water with circular frequency equal to the encounter frequency ω_E, given by the equation:

$$\omega_E = |\omega - ku\cos\beta|, \tag{19}$$

where ω is the circular frequency of regular wave, $k \equiv 2\pi/\lambda$ is the wave number, λ is the wave length, u is the forward speed of the ship, and β is the heading angle between the ship speed and direction of wave propagation. And 2), predicting the hydrodynamic coefficients from the solution of equations of conservation for the viscous flow, including the boundary layer. As shown by Weymouth *et al*. (2005), such a problem is extremely complex. The thickness of the boundary layer varies in the course of motion, and outside the boundary layer – as shown earlier – flow is irrotational, i.e. potential. Hence, turbulence occurs solely inside the boundary layer, while outside it – the flow is potential.

The first approach ignores simply the boundary layer. Here arises the question, if the hydrodynamic coefficients could be determined accounting for the boundary layer, but not in the so complicated way, as discussed by Weymouth *et al*. (2005). It seems there is such a possibility. Note that 1) outside the boundary layer flow is potential, and 2) the boundary layer moves with the ship, as if being fixed to it. Hence, we can assume that the mass of the ship is augmented by the mass of the boundary layer, while the hydrodynamic forces are such, as if the wetted surface of the ship coincided with the outer surface of the boundary layer. Hence, the key meaning in this approach has the thickness δ of the turbulent boundary layer. As first approximation, it could be taken the same as for a flat plate:

$$\delta = 0.37\left(\frac{\nu}{ux}\right)^{1/5} x, \tag{20}$$

where ν is the kinematic viscosity of water, u is the forward speed of the ship, and x is distance from the forward end of the ship below water. The fraction inside the parentheses is the inverse of the Reynolds number, related to x. Behind the separation point, the thickness of the boundary layer can be taken as $\delta = 0$. Numerical values of δ are shown in Table. 1 for the coefficient of viscosity $\nu = 10^{-6}$ m^2/s, and two vessel's speeds $u = 5$ m/s and 10 m/s.

Table. 1. Thickness of turbulent boundary layer

	$u = 5$ m/s		$u = 10$ m/s	
x [m]	Re	δ [m]	Re	δ [m]
10	$5.0\cdot10^7$	0.107	$1.0\cdot10^8$	0.093
20	$1.0\cdot10^8$	0.186	$2.0\cdot10^8$	0.162
50	$2.5\cdot10^8$	0.387	$5.0\cdot10^8$	0.337
100	$5.0\cdot10^8$	0.674	$1.0\cdot10^9$	0.586
200	$1.0\cdot10^9$	1.173	$2.0\cdot10^9$	1.021

It is worth realising that accurate prediction of the hydrodynamic coefficients is not required for the satisfactory prediction of ship motion in a seaway. Coefficients of stiffness c_{33}, c_{44}, c_{55} are at least an order of magnitude larger (the first and third are usually more than two orders larger) than the added mass and damping coefficients. The overwhelming dominance of hydrostatic stiffness and highly linear wave excitation allow the strip theory to predict ship motions with reasonably high degree of accuracy. Things are improved further by the counteracting non-linear effects of hydrodynamic coefficients a_{44} and b_{44} (Figure. 1).

5. CASE STUDIES

To shed some light on the effect of viscosity in ship dynamics it is worth recalling four case studies, known in literature: 1) sliding oscillations of a flat surface, 2) angular oscillations of a cylinder, 3) sliding oscillations of a cylinder, and 4) oscillations of an arbitrary body.

5.1 SLIDING OSCILLATIONS OF A FLAT SURFACE

This is a classic problem, discussed in almost every textbook on fluid mechanics, solved by Stokes in 1851. Consider an infinite surface at the x-plane, performing harmonic oscillations in the z direction with the velocity $u = u_0\cos\omega t$. By assuming that the field of velocity $\boldsymbol{v} = v\boldsymbol{k}$ is of laminar type, i.e. it has one component only in the direction of the z-axis, where $v = v(x)$ is a function of x (distance from the plane), N–S equation (2) reduces to two scalar equations: $p = const$, and the equation of diffusion:

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial^2 x}, \tag{21}$$

Its solution is as follows:

$$v = u_0 e^{-kx} \cos(kx - \omega t), \tag{22}$$

where $k = (\omega/2\nu)^{1/2}$ is the wave number. The inverse of the wave number, denoted by $\delta \equiv 1/k = (2\nu/\omega)^{1/2}$ is the *depth of penetration*, known better as the *skin depth*. At a distance $x = 3\delta$ the velocity drops to $e^{-3} \approx 5\%$ of the value at the oscillating surface. And at a distance $x = 5.3\,\delta$ it drops to 0.5%. We can assume then that the thickness of the boundary layer equals ~5δ. The depth of penetration increases with the kinematic viscosity ν and *decreases* with the circular frequency ω. For a finite plate, the depth of penetration is presumably much thinner.

For example, for water the kinematic coefficient of viscosity $\nu \approx 10^{-6}$ m^2/s, and for air $\nu \approx 15{\cdot}10^{-6}$ m^2/s. Assuming the circular frequency of oscillations ω = 1 rad/s, the depth of penetration for water equals δ = 1.4 mm, and for air δ = 5.5 mm. At a distance 5δ, i.e. 7 mm for water, and 27 mm for air, the fluid is practically at rest, despite the oscillations. These quantities are inversely proportional to √ω. Real frequencies occurring in seakeeping are from the range $\omega \in \langle 0.2,\ 4\rangle$ rad/s. Hence, the skin depth is small. Note that this quantity value is at least two orders of magnitude smaller than the values for a stationary boundary layer (Table. 1).

Knowing the velocity field, the tangential stress on the surface can be found by the equation $\tau = \mu\partial v/\partial x$. Substituting $x = 0$, the following is obtained for the stresses:

$$\tau = -\mu k u_0(\cos\omega t - \sin\omega t),$$

where $\mu k = (½\rho\mu\omega)^{1/2}$. Hence, a phase shift exists between stress and speed. Since the acceleration of the surface $\dot{u} = -\omega u_0 \sin\omega t$, the above can be written as:

$$\tau = -(\mu k/\omega)\dot{u} - \mu k u = -(½\rho\mu/\omega)^{1/2}\dot{u} - (½\rho\mu\omega)^{1/2} u \equiv -m\dot{u} - Nu, \tag{23}$$

where $m = (½\rho\mu/\omega)^{1/2}$ is the added mass per unit area, while $N = (½\rho\mu\omega)^{1/2}$ is the coefficient of damping per unit area. If the fluid is on both sides of the surface, the expressions for m and N should be doubled. Both components of the stress, which are the same as the frictional resistance per unit area, are directed against the appropriate parameters of motion.

Friction is associated with dissipation of energy. The said quantity can be obtained as work of friction forces. Dissipation of energy per unit time and unit area is equal to mean value of the product of the tangential stress and the speed of the surface:

$$\langle \tau u\rangle = -{}^1/_T\int_0^T (m\dot{u}u + Nu^2)dt,$$

where T is the period of oscillations. Since $\dot{u}udt = d½u^2$, the first term provides no contribution due to the oscillations of velocity. A contribution provides the other term, equal to: $\langle \tau u\rangle = -½Nu_0^2$.

As can be seen, the coefficient of damping $N = (½\rho\mu\omega)^{1/2}$ is responsible for dissipation, not the added mass. This can be taken as a general rule.

5.2 ANGULAR OSCILLATIONS OF A CYLINDER

Consider now the velocity field around an infinitely long cylinder of radius r_0, performing angular oscillations around its axis in infinite fluid with a circular frequency ω. The z-axis coincides with the axis of the cylinder (Figure. 2).

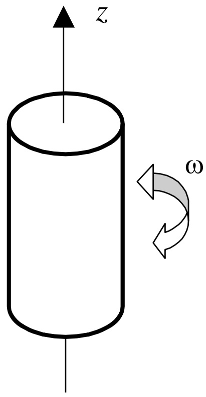

Figure. 2

It is assumed again that the velocity field is of laminar type, i.e. it has only a circumferential component $\boldsymbol{v} = v\boldsymbol{e}_\theta$, a value of which depends on the co-ordinate r (distance from the axis of the cylinder) and on time t, i.e. $v = v(r, t)$. This component is constant for a given r at a fixed time instant, which results from the equation of continuity. Similarly, the pressure field $p = p(r, t)$. Note that the problem is 2-D, therefore no quantity can depend on the variable z.

With these assumptions grad$p$ has only a radial component. Similarly, the convective acceleration $(v\cdot\nabla)v$ has solely a radial component, equal to the centrifugal acceleration. The two remaining terms in equation of motion (2), i.e. the local acceleration and Laplacian of velocity have only circumferential components, as $\Delta(v\boldsymbol{e}_\theta) = (v'' + v'/r - v/r^2)\boldsymbol{e}_\theta$, where ' means differentiation with respect to r. Equation (2) yields then two scalar equations, one for the pressure field $dp/dr = \rho v^2/r$, and the other for the velocity field, which has a form of the equation of diffusion:

$$\partial v/\partial t = \nu(v'' + v'/r - v/r^2). \qquad (24)$$

As can be seen, the pressure field $p = p(r, t)$ is obtained by integrating the centrifugal accelerations, dependent on the instantaneous velocity field. On the other hand, the velocity field $v = v(r, t)$ is determined by decoupled equation (24).

Determination of the velocity field around an oscillating cylinder is not as easy as for an oscillating plane. Equation (24) suggests that we can seek its solution by a separation of variables. Namely, we can postulate that $v = u_0e^{-i\omega t}R$, where $R = R(r)$ is a function of variable r only, whereas $v = u_0e^{-i\omega t}$ is the velocity on the surface of the cylinder at $r = r_0$.

Substituting v to equation (24) yields a characteristic equation for the function R:

$$-i\omega R = \nu(R'' + R'/r - R/r^2),$$
$$\nu(R'' + R'/r - R/r^2) + i\omega R = 0,$$
$$R'' + R'/r - R/r^2 + (i\omega/\nu)R = 0.$$

Introducing notation $K^2 \equiv i\omega/\nu$, the above equation will take the form:

$$R'' + R'/r + (K^2 - {}^1/_{r^2})R = 0.$$

This is a Bessel equation of the first order [the Bessel equation of the n-th order is defined as: $R'' + R'/r + (K^2 - {}^{n^2}/_{r^2})R = 0$]. Multiplying it by r^2 yields the characteristic equation in an equivalent form:

$$r^2R'' + rR' + (K^2r^2 - 1)R = 0.$$

Introducing a nondimensional radius $\rho = Kr$, the function R becomes a function $R = R(\rho)$. The characteristic equation will become a normalised Bessel equation of the first order:

$$R'' + R'/\rho + (1 - {}^1/_{\rho^2})R = 0, \qquad (25)$$

defining the function $R(\rho)$, where ' means now differentiation with respect to ρ. Its solution are cylindrical functions of the first and second kind $J_1(\rho)$ and $N_1(\rho)$. The asymptotic form of the two functions for large arguments $\rho = Kr$ is as follows (Abramowitz & Stegun, 1970):

$$J_1 = (2/\pi\rho)^{1/2}\sin(\rho - \tfrac{1}{4}\pi),$$
$$N_1 = -(2/\pi\rho)^{1/2}\cos(\rho - \tfrac{1}{4}\pi). \qquad (26)$$

Graphs of the functions J_1 and N_1 for a real argument ρ are shown in Figure. 3. For $\rho > \sim 3$ they can be replaced the asymptotic approximations.

The parameter K is given by the equation: $K^2 \equiv i\omega/\nu$. Hence, $K = k'\sqrt{i}$, where $k' = (\omega/\nu)^{1/2}$. Thus, the nondimensional variable $\rho \equiv Kr = k'r\sqrt{i} \equiv \rho'\sqrt{i}$ is a complex number, where $\rho' \equiv k'r = |\rho|$ is the module of the variable ρ. Considering that $\sqrt{i} = \pm(1 + i)/\sqrt{2}$, the variable ρ can be presented in the equivalent form:

$$\rho = \pm(1 + i)k'r/\sqrt{2} = \pm(kr + ikr), \qquad (27)$$

where $k = k'/\sqrt{2} = (\omega/2\nu)^{1/2}$.The choice of the sign depends on the condition in infinity. We will see later that the sign should be negative, i.e. corresponding to the phase $-\tfrac{3}{4}\pi$.

Real and imaginary parts of the cylindrical functions can be obtained by substituting $\rho = \rho'\sqrt{i}$ in the polynomial expansions; they are functions of the amplitude $\rho' \equiv k'r = \sqrt{2}kr$ and are called the *Kelvin* (*Thomson*) functions.

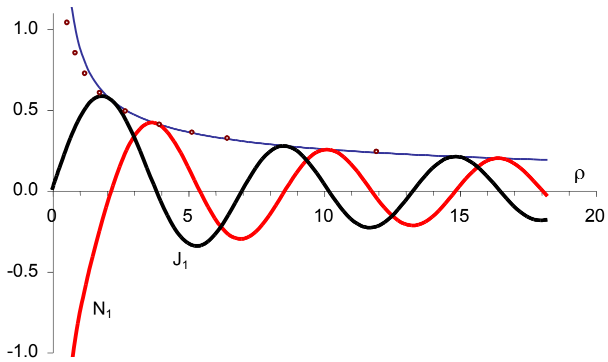

Figure. 3. Runs of cylindrical functions J_1, N_1 and the amplitude $(J_1^2 + N_1^2)^{1/2}$ for real ρ

The number $\sqrt{i}$ has two phases $-\tfrac{3}{4}\pi$ and $\tfrac{1}{4}\pi$. In electrical engineering, for analysing the so-called *skin effect*, the variable $\rho = \rho'\sqrt{(-i)}$ is of importance, which has two phases: $\tfrac{3}{4}\pi$ and $-\tfrac{1}{4}\pi$. Skin effect is the tendency of an alternating electric current to become distributed within a conductor in such a way that the current density is largest near the surface of the conductor, and decreases towards its centre. The electric current flows mainly at the conductor "skin", between the outer surface and a level called the skin depth δ. This effect increases effective resistance of the conductor at higher frequencies where the skin depth is smaller, thus reducing the effective cross-section of the conductor. Hence, the determination of current density in the conductor is an *inner* problem, contrary to the determination of the velocity field outside an oscillating cylinder. The inner problem in fluid mechanics means determination of the velocity field *inside* the cylinder.

The Kelvin functions solely concern the variable $\rho = \rho'e^{\pm\frac{3}{4}\pi i}$, with phases $\pm\tfrac{3}{4}\pi$. They are denoted as follows:

$$J_1(\rho'e^{\pm\frac{3}{4}\pi i}) \equiv \mathrm{ber}_1\,\rho' \pm i\,\mathrm{bei}_1\,\rho',$$
$$N_1(\rho'e^{\pm\frac{3}{4}\pi i}) \equiv \mathrm{yer}_1\,\rho' \pm i\,\mathrm{yei}_1\,\rho'.$$

When a complex argument ρ tends to infinity, cylindrical functions become unbounded. For a real variable, they present damped oscillations, as in Figure. 3.

A general solution of equation (25) is a linear combination of the fundamental solutions, i.e. $R(\rho) = \alpha J_1(\rho) + \beta N_1(\rho)$, where α and β are arbitrary constants. We have to find a combination where the function $R(\rho)$ decays with growth of the complex argument ρ. To ease the answer to this question, consider the asymptotic form of the two functions, given by equation (26). Substituting $\rho = x + \mathrm{i}y$, the following is obtained:

$$J_1 = (2/\pi\rho)^{1/2}[\cosh y \sin(x - \tfrac{1}{4}\pi) + \mathrm{i}\sinh y \cos(x - \tfrac{1}{4}\pi)],$$
$$N_1 = -(2/\pi\rho)^{1/2}[\cosh y \cos(x - \tfrac{1}{4}\pi) - \mathrm{i}\sinh y \sin(x - \tfrac{1}{4}\pi)],$$

where $x = y = kr$. When the imaginary part y tends to $\pm\infty$, both functions grow to infinity. Create then a new cylindrical function called the *Hankel function*: $H_1 \equiv J_1 - \mathrm{i}N_1$, denoted in literature by $H_1^{(2)}$. After simplifications, the following is obtained:

$$H_1 = (2/\pi\rho)^{1/2} e^{y}[\sin(x - \tfrac{1}{4}\pi) + \mathrm{i}\cos(x - \tfrac{1}{4}\pi)].$$

The Hankel function vanishes in infinity in the lower half-domain, when $y < 0$. This happens, when the nondimensional variable ρ, given by equation (27), is taken with the minus sign: $\rho = -(kr + \mathrm{i}kr) = -\sqrt{2}kre^{\mathrm{i}\pi/4} = -\rho' e^{\mathrm{i}\pi/4} = \rho' e^{-\frac{3}{4}\pi\mathrm{i}}$, which gives the phase $-\frac{3}{4}\pi$. Considering that $x = y = -kr$, after simplifications we get an asymptotic form of the Hankel function:

$$H_1(kr) = -2^{1/4}(\pi kr)^{-1/2} e^{-kr} e^{\mathrm{i}(kr + \frac{1}{8}\pi)} = M_1 e^{\mathrm{i}(kr + \frac{1}{8}\pi)}, \qquad (28)$$

where $M_1 = -2^{1/4}(\pi kr)^{-1/2}$ is the amplitude of the function $H_1(kr)$. We will see that the sign is unimportant. For a finite argument ρ the expression for the Hankel function $H_1(kr)$ is far more complicated. It can be obtained from the polynomial expansions of the cylindrical functions. For a variable $\rho = \rho' e^{\pm\frac{3}{4}\pi\mathrm{i}}$, the real and imaginary parts are denoted, as below:

$$H_1(\rho' e^{\pm\frac{3}{4}\pi\mathrm{i}}) \equiv \mathrm{her}_1(\rho') \mp \mathrm{i}\,\mathrm{hei}_1(\rho').$$

These functions are related in a simple way to modified Bessel functions: $\mathrm{ker}_1\rho' \equiv -\frac{1}{2}\pi\mathrm{hei}_1\rho'$ and $\mathrm{kei}_1\rho' \equiv \frac{1}{2}\pi\mathrm{her}_1\rho'$. And the modified Bessel functions are related in turn to the first derivatives of the functions $\mathrm{ker}\,\rho'$ and $\mathrm{kei}\,\rho'$ of the zero order:

$$\mathrm{ker}_1\rho' = (\mathrm{ker}'\rho' - \mathrm{kei}'\rho')/\sqrt{2},$$
$$\mathrm{kei}_1\rho' = (\mathrm{ker}'\rho' + \mathrm{kei}'\rho')/\sqrt{2}.$$

Hence, to calculate the real and imaginary parts of the Hankel function $H_1(\rho)$ it is sufficient to know the derivatives of the Hankel functions of the zero order $\mathrm{ker}'\rho'$ and $\mathrm{kei}'\rho'$. They are given by effective polynomial approximations (Abramowitz & Stegun, 1970). Their graphs are shown in Figure. 4. For large values of $\rho' \equiv k'r = \sqrt{2}kr$, both functions oscillate, passing through the same zeros.

With the increase of the module of the nondimensional radius $\rho' = |\rho|$, decrement of damping of the amplitude of the Hankel function H_1 increases monotonically from $-\infty$ to an asymptotic value $-1/\sqrt{2} \approx -0.707$ (Figure. 5). It means that for large enough $\rho' > \sim 4$ the velocity field vanishes as $(\rho'/\sqrt{2})^{-1/2} e^{-\rho'/\sqrt{2}}$. In other words, the variable ρ' becomes the variable $\rho'/\sqrt{2} = kr$. The solution must be then expressed in terms of the variable kr, as seen in equation (28).

Now, a general solution of equation (25) is a function $R(\rho) = \alpha H_1(\rho)$. It fulfils the condition in infinity, as it vanishes to zero. The constant α is chosen from the kinematic condition on the surface of cylinder $\rho = \rho_0$ to be equal to 1. Hence, $\alpha = 1/H_1(\rho_0)$. The velocity field is given then by the equation: $v = u_0 e^{-\mathrm{i}\omega t} H_1(\rho)/H_1(\rho_0)$. Taking the real part, the following is obtained:

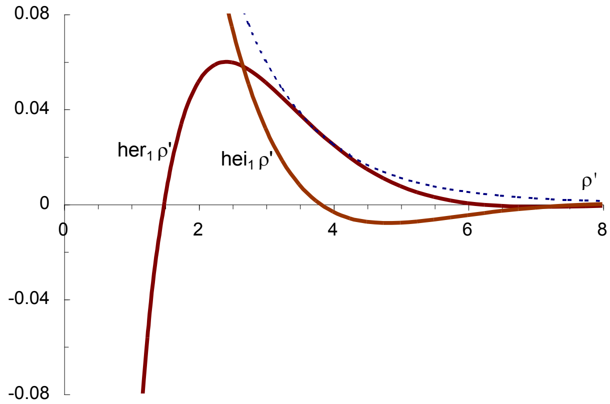

Figure. 4. Run of real and imaginary parts of Hankel function H_1 and their amplitude

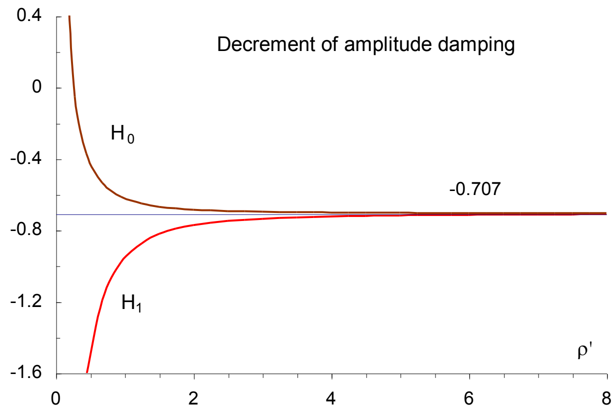

Figure. 5. Decrement of amplitude damping of Hankel function H_0 and H_1

$$v = u_0[(AA_0 + BB_0)\cos\omega t + (AB_0 - A_0B)\sin\omega t]/(A_0^2 + B_0^2), \qquad (29)$$

where $A \equiv \mathrm{her}_1(\rho')$ and $B \equiv \mathrm{hei}_1(\rho')$ are the real and imaginary parts of the Hankel function $H_1(\rho)$, A_0 and B_0 are the values on the cylinder surface. On the surface of cylinder $r = r_0$, the above equation yields $v = u_0\cos\omega t$. As can be seen from Figure. 4, for $\rho' < \sim 8$ the functions A and B are not of oscillatory character, therefore the solution has the character of a standing wave.

Equation (29) can be largely simplified for arguments $\rho' \equiv k'r > \sim 8$, which happens, when the radius $r > r_{min} \equiv \sim 8/k' = 8/\sqrt{2}k = 4\sqrt{2}/k = 4\sqrt{2}\delta \approx 5.7\delta$. For water, for circular frequency $\omega = 1$ rad/s, $r_{min} = \sim 8$ mm, and for air $r_{min} = \sim 32$ mm. The quantity r_{min} is inversely proportional to $\sqrt{\omega}$; it is then generally small. Resorting to equation (28), it is easy to find the real part of the velocity $v = u_0 e^{-i\omega t} H_1(\rho)/H_1(\rho_0)$:

$$v = u_0 (r_0/r)^{1/2} e^{-k(r-r_0)} \cos[k(r-r_0) - \omega t] = \\ = u_0 (r_0/r)^{1/2} e^{-kh} \cos(kh - \omega t),$$

valid for $r > r_{min}$, where $h \equiv r - r_0$ is the distance from the cylinder surface, and $k = (\omega/2\nu)^{1/2}$. Expressing the ratio of radii in terms of h, we get eventually:

$$v = \frac{u_0}{\sqrt{1+\frac{h}{r_0}}} e^{-kh} \cos(kh - \omega t)\cdot \quad (30)$$

For $h > \sim 5\delta$, $v \approx 0$. If r_0 grows indefinitely, the velocity field tends to the field of a flat surface, given by equation (22), with $x \equiv h$. The velocity field in the case of a cylinder vanishes somewhat faster than for a flat surface. However, the depth of penetration $\delta = 1/k$ is the same in both cases. When $r < r_{min}$, i.e. for cylinders with small radii, the velocity field has to be found using equation (29), which requires the knowledge of the Hankel function $H_1(\rho)$.

It can be shown that the error, with which the Hankel function $H_1(kr)$, given by equation (28), fulfils Bessel equation (25), very quickly decreases with a growth of the nondimensional radius kr. The absolute value of the error equals $\frac{3}{4} \cdot 2^{1/4} (\pi kr)^{-1/2} e^{-kr}$. For $kr = 2, 3, 4$ the error equals merely 4.8%, 1.4%, and 0.5%.

When the cylinder rotates in one direction with a constant angular velocity ω, it induces a stationary velocity field, as for a rectilinear vortex, given by the equation:

$$v = u_0 r_0/r = u_0/(1 + h/r_0), \quad (31)$$

where u_0 is the velocity on the surface of the cylinder at $r = r_0$. As can be seen, the velocity field has now a completely different character than in the case of an oscillating cylinder. 1°, despite viscosity, the velocity field is potential and independent of viscosity. 2°, if r_0 grows indefinitely, it tends to a uniform flow $v = u_0$, with an infinitely thick boundary layer. 3°, it decays much slower than for an oscillating cylinder. The velocity drops to $1/n$ of the value u_0 at the surface at a distance $h = (n - 1)r_0$, independent of viscosity. For instance, it drops to 0.5% = $^1/_{200}$, at a distance $h = 199r_0$, extremely large in comparison to an oscillating cylinder. By the sheer fact that the body oscillates, the boundary layer reduces to amazingly small dimensions.

5.3 SLIDING OSCILLATIONS OF A CYLINDER

Consider now an infinitely long cylinder of radius r_0, performing sliding oscillations along its axis in infinite fluid with a circular frequency ω (Figure. 6). It is assumed again that the velocity field has only a longitudinal component $\boldsymbol{v} = v\boldsymbol{e}_z$, which can be solely a function of the co-ordinate r (distance from the axis of the cylinder) and on time t, i.e. $v = v(r, t)$. This results from the equation of continuity, symmetry of the problem and its 2-D character.

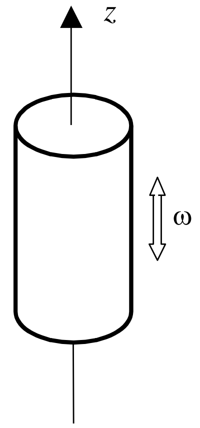

Figure. 6.

With these assumptions, the convective acceleration $(v\cdot\nabla)v$ vanishes, and equation (2) reduces to two scalar equations: i.e. $p = const$, and the diffusion equation: $\partial v/\partial t = \nu\Delta v$, where the Laplacian $\Delta v = (rv')'/r = v'' + v'/r$ (the sign ' means differentiation with respect to r). Hence:

$$\partial v/\partial t = \nu(v'' + v'/r). \quad (32)$$

We can postulate, as before, that the solution of equation (32) is of the form: $v = u_0 e^{-i\omega t} R$, where $R = R(r)$ is a function of the variable r only, whereas $v = u_0 e^{-i\omega t}$ is the velocity of the cylinder. Substituting v in equation (32) yields a characteristic equation for the function R:

$$-i\omega R = \nu(R'' + R'/r), \\ R'' + R'/r + (i\omega/\nu)R = 0.$$

Introducing the notation $K^2 \equiv i\omega/\nu$, the above will take the form: $R'' + R'/r + K^2 R = 0$. Introducing the nondimensional radius $\rho = Kr$, the function R becomes a function $R = R(\rho)$, for which the characteristic equation is as follows:

$$R'' + R'/\rho + R = 0, \quad (33)$$

where ' means now differentiation with respect to ρ. This is a Bessel equation of the zero order, defining the function $R = R(\rho)$. Its solution are cylindrical functions of the first and second kind $J_0(\rho)$ and $N_0(\rho)$. At zero these functions have values as follows: $J_0(0) = 1$, $N_0(0) = -\infty$. The asymptotic form of the two functions for large arguments $\rho = Kr$ is as follows (Abramowitz, Stegun, 1970):

$$J_0 = (2/\pi\rho)^{1/2}\cos(\rho - \tfrac{1}{4}\pi),\quad N_0 = -(2/\pi\rho)^{1/2}\sin(\rho - \tfrac{1}{4}\pi). \qquad (34)$$

Graphs of the functions J_0 and N_0 for a real argument ρ are shown in Figure. 7. For $\rho > \sim 1$ they can be replaced the asymptotic approximations.

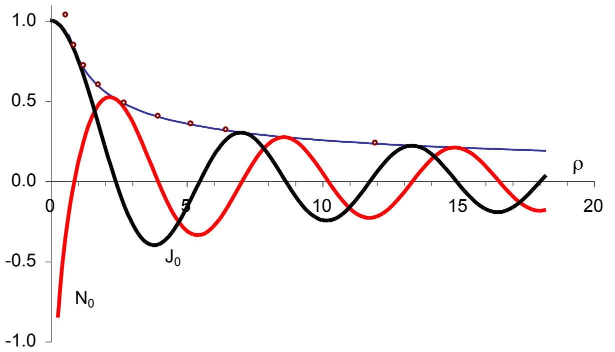

Figure. 7. Runs of cylindrical functions J_0, N_0 and the amplitude $(J_0^2 + N_0^2)^{1/2}$ for real ρ

The non-dimensional radius ρ is a complex quantity, given by equation (27). Substituting to the Bessel functions $\rho = \rho' e^{\pm\frac{3}{4}\pi i}$, where $\rho' \equiv \sqrt{2}kr$, we get

$$J_0(\rho' e^{\pm\frac{3}{4}\pi i}) \equiv \mathrm{ber}\,\rho' \pm i\,\mathrm{bei}\,\rho',\quad N_0(\rho' e^{\pm\frac{3}{4}\pi i}) \equiv \mathrm{yer}\,\rho' \pm i\,\mathrm{yei}\,\rho'.$$

When the complex argument ρ tends to infinity, both functions become unbounded. For a real variable, they present damped oscillations, as in Figure. 7. As before, the Hankel function $H_0 \equiv J_0 + iN_1$, denoted in literature by $H_0^{(1)}$ (note the change of the sign), vanishes in infinity in the lower half-domain, when $y < 0$. Its asymptotic expansion for large ρ is as follows:

$$H_0 = (2/\pi\rho)^{1/2} e^{y}[\cos(x - \tfrac{1}{4}\pi) - i\sin(x - \tfrac{1}{4}\pi)].$$

This happens, when the nondimensional variable $\rho = \rho' e^{\pm\frac{3}{4}\pi i}$ is taken with the phase $-\frac{3}{4}\pi$. Considering that $x = y = -kr$, the asymptotic form of the Hankel function is as follows:

$$H_0(kr) = 2^{1/4}(\pi kr)^{-1/2} e^{-kr} e^{i(kr + \frac{5}{8}\pi)} = M_0 e^{i(kr + \frac{5}{8}\pi)}, \qquad (35)$$

where $M_0 = 2^{1/4}(\pi kr)^{-1/2}$ is the amplitude of the function $H_0(kr)$. For a finite argument ρ the expression for the Hankel function $H_0(kr)$ can be obtained from the polynomial expansions of the cylindrical functions. For a variable $\rho = \rho' e^{\pm\frac{3}{4}\pi i}$, the real and imaginary parts are denoted, as:

$$H_0(\rho' e^{\pm\frac{3}{4}\pi i}) \equiv \mathrm{her}(\rho') \pm i\,\mathrm{hei}(\rho'),$$

where the real and imaginary parts of the function $H_0(\rho' e^{\pm\frac{3}{4}\pi i})$, i.e. $\mathrm{her}(\rho')$ and $\mathrm{hei}(\rho')$, shown in Figure. 8, are calculated with the help of the function $\mathrm{ker}\,\rho' \equiv -\frac{1}{2}\pi\,\mathrm{hei}(\rho')$ and $\mathrm{kei}\,\rho' \equiv \frac{1}{2}\pi\,\mathrm{her}\,\rho'$; the functions $\mathrm{ker}\,\rho'$ and $\mathrm{kei}\,\rho'$ are given by effective polynomial approximations (Abramowitz & Stegun, 1970). For large values of $\rho' \equiv k'r = \sqrt{2}kr$, both functions oscillate, passing through the same zeros.

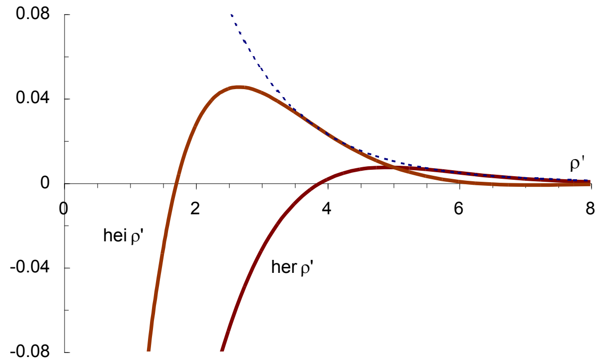

Figure. 8. Runs of real and imaginary parts of Hankel function H_0 and their amplitude

As before, with the increase of the nondimensional radius $\rho' = |\rho|$, decrement of damping of the amplitude of the Hankel function H_0 decreases monotonically from $+\infty$ to an asymptotic value $-1/\sqrt{2} \approx -0.707$ (Figure. 5). It means that for large $\rho' > \sim 2$ the velocity field vanishes as $(\rho'/\sqrt{2})^{-1/2} e^{-\rho'/\sqrt{2}}$. In other words, the variable ρ' becomes the variable $\rho'/\sqrt{2} = kr$. The solution must be then expressed in terms of the variable kr, as seen in equation (35).

A general solution of equation (33) is a function $R(\rho) = \alpha H_0(\rho)$. It fulfils the condition in infinity, as it vanishes to zero. The velocity field is given by the equation: $v = u_0 e^{-i\omega t} H_0(\rho)/H_0(\rho_0)$. Taking the real part, we get:

$$v = u_0[(AA_0 + BB_0)\cos\omega t - (AB_0 - A_0B)\sin\omega t]/(A_0^2 + B_0^2), \qquad (36)$$

where $A \equiv \mathrm{her}_1(\rho')$ and $B \equiv \mathrm{hei}_1(\rho')$ are the real and imaginary parts of the Hankel function $H_1(\rho)$, A_0 and B_0 are the values on the cylinder. As can be immediately checked, on the surface of cylinder $r = r_0$, the above equation yields $v = u_0\cos\omega t$. For $\rho' < \sim 8$ the functions A and B have no oscillatory character (see Figure. 8), therefore the solution has the character of a standing wave.

Equation (36) can be largely simplified for arguments $\rho' \equiv k'r > \sim 8$, which happens, when the radius $r > r_{min} \equiv \sim 8/k' = 8/\sqrt{2}k \approx 5.7\delta$. Resorting to equation (35), it is easy to find the real part of the velocity $v = u_0 e^{-i\omega t} H_1(\rho)/H_1(\rho_0)$, reducing to equation (30), the same as for angular oscillations, valid for $r > r_{min} \approx 5.7\delta$. For $h > \sim 5\delta$, $v \approx 0$.

It is interesting that in the three case studies the boundary layer is the same and at least two orders of magnitude smaller than the values for a stationary boundary layer

(Table. 1). Further, the velocity field is found from equation of diffusion (6), without resorting to the idea of vorticity. This feature of the field is a resultant quantity, having no impact on the calculations.

5.4 OSCILLATIONS OF AN ARBITRARY BODY

As we could see in the foregoing three case studies, contrary to stationary flows the boundary layer is thin on oscillating bodies. It can be shown that the same holds for an arbitrary oscillating body. A proof is provided below.

In the case of sliding oscillations of a flat surface or a cylinder the convective acceleration $(v\cdot\nabla)v$ vanishes everywhere. In the case of an arbitrary surface, this no longer holds. In linear problems, non-linear terms are ignored by definition, irrespective of their size. We are entitled to do it, if they are small in relation to the remaining terms in the governing equations.

The convective acceleration $(v\cdot\nabla)v$ is the only non-linear term in N–S equation (2). It is relatively easy to assess its magnitude (Landau & Lifszic, 2009). The operator $(v\cdot\nabla)$ means differentiation along the direction of velocity. In close vicinity of the body the velocity is basically parallel to its surface. Therefore $(v\cdot\nabla)v \sim v^2/l$, where l is a characteristic dimension of the body. In an oscillatory motion the velocity $v \sim \omega A$, where A is the amplitude of oscillations. Hence,

$$(v\cdot\nabla)\,v \sim (\omega A)^2/l.$$

On the other hand, the following holds for the local acceleration:

$$\partial \boldsymbol{v}/\partial t \sim \omega v \sim \omega^2 A.$$

Comparing the two expressions, we get $(v\cdot\nabla)v << \partial v/\partial t$, if $A << l$, i.e. if the amplitude of oscillations is small in relation to the size of the body. In addition, it is easy to show that the terms $\partial \boldsymbol{v}/\partial t$ and $\nu\Delta \boldsymbol{v}$ are of the same magnitude.

The Reynolds number is normally defined for stationary flows, when a body moves at constant speed. For oscillatory motion this number is defined as $\mathrm{Re} = \omega A l/\nu$. In linear problems amplitudes of motion are assumed to be small, as we say – infinitely small. The Reynolds numbers are, therefore, small by definition, leaving little room for turbulence, if at all.

With the help of the above considerations, some properties of motion can be deduced from the linearised equation (5). The operator of rotation (curl) can be applied to both its sides. As the rotation of the gradient vanishes, and introducing notation $\Omega \equiv \mathrm{rot}v$ for the vorticity, we get:

$$\partial/_{\partial t}\Omega = \nu\Delta\Omega, \qquad (37)$$

i.e. Ω fulfils the equation of diffusion (6). It follows from the foregoing that such an equation leads to exponential decay of the quantity described by it, in this case the vorticity. In other words, the motion of the fluid induced by oscillating body is rotational in some layer around the body. Vorticity decays rapidly with a distance from the body, turning at some distance to a potential flow, despite viscosity. The depth of penetration of the vorticity is identical, as for the velocity, equal to $\delta = 1/k = (2\nu/\omega)^{1/2}$.

The quantity δ can be large or small in relation to the body. The case of $\delta \gg l$ occurs, if $\omega l^2 \ll \nu$, i.e. when oscillations are extremely slow, far below the range of interest. In such cases changes of velocity are very slow. The motion of the fluid is therefore quasi-stationary. That is to say, at each time instant fluid motion is the same as for a uniform motion of the body with the speed at given time instant. The boundary layer as such stretches practically over the entire domain.

The opposite case $\delta << l$, i.e. of *thin* boundary layer occurs, when $\omega l^2 >> \nu$. As ν is small, this occurs practically at the entire range of frequency of oscillations ω that are of interest. In seakeeping, it is from the range $\omega \in \langle 0.2, 4\rangle$ rad/s. As the boundary layer is thin the effect of viscosity on the hydrodynamic coefficients is generally minimal.

6. CONCLUSIONS

Based on the results and arguments presented in this work the following conclusions can be drawn:

- it is possible to account for viscosity in the hydrodynamic coefficients (added mass and damping coefficient) but this should not be done by the RANS equations employing turbulence models, as in linear problems there is no room for turbulence
- hydrodynamic coefficients for roll can include the effect of viscosity through the linearised Navier–Stokes equations that do not need any turbulence models. For other degrees of freedom the effect of viscosity is vestigial
- contrary to stationary flows, the boundary layer on oscillating ships is thin; therefore its effect on the hydrodynamic coefficients can only be of secondary meaning
- viscosity opens room for memory effects even in an unbounded domain
- commercial RANS software can be used in seakeeping for solving the Navier–Stokes equations provided no turbulence stresses are assumed
- advanced turbulence models should be based on modelling Reynolds stresses

7. REFERENCES

1. ABRAMOWITZ M., STEGUN J. A. *Handbook of mathematical functions*, Dover Publications, Inc., New York 1970, 9th edition

2. DUDZIAK J. *Teoria okrętu*, Wydawnictwo Morskie, Gdańsk, 1988
3. FALTINSEN O. M. *Sea loads on ships and offshore structures*, Cambridge University Press, 1990
4. IKEDA Y., HIMENO Y., TANAKA N. *A prediction method for ship roll damping*, Dept. of Naval Architecture, University of Osaka Prefecture, Report 00405, 1978
5. LANDAU L., LIFSZIC E. *Hydrodynamika*, Wydawnictwo Naukowe PWN, Warszawa 2009
6. McDONOUGH J. M. *Introductory lectures on turbulence: physics, mathematics and modeling*, Dept. of Mechanical Engineering and Mathematics, University of Kentucky, 2007, http://web.engr.uky.edu/~acfd/lctr -notes634.pdf
7. PAWŁOWSKI M. *Approximation of the non-linear roll damping*, Proc., 11th Intl. Ship Stability Workshop, Wageningen, 2010, 7 pp., http://shipstab.org/index.php /conference-workshop-proceedings
8. QUÉRARD A.B.G., TEMAREL P., TURNOCK S.R. *Hydrodynamics of ship-like sections in heave, sway and roll motions predicted using an unsteady RANS method*, Proc. IMechE, Vol. 223, Part M: J. Engineering for the Maritime Environment, No 2/2009, p.227–238, DOI 10.1243/14750902JEME114
9. SALUI K. B., SARKAR T., VASSALOS D. *An improved method for determining hydrodynamic coefficients in roll motion using CFD techniques*, Ship Technology Research, Vol. 47, 2000, pp. 161–174
10. SALVESEN N., TUCK, E. O., FALTINSEN O.: *Ship motions and sea loads*, Trans. Society of Naval Architects and Marine Engineers, Vol. 78, pp 250–279, Discussion: 279–287 (1970)
11. SPOUGE J. R. *Non-linear analysis of large amplitude rolling experiments*, Intl. Shipbuilding Progress, Vol. 35, No. 403, 1988, pp. 271-320
12. VUGTS J. H *The hydrodynamic coefficients for swaying, heaving and rolling cylinders in a free surface*, Intl. Shipbuilding Progress, No. 167, Vol. 15, 1968, pp. 251–276
13. WEYMOUTH G. D., WILSON R. V., STERN F. *RANS computational fluid dynamics predictions of pitch and heave motions in head seas*, Journal of Ship Research, Vol. 49, No. 2, June 2005, pp. 80–97

