THE STABILITY OF A FREELY FLOATING SHIP

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SUMMARY

The paper presents the problem of calculating the righting arms (*GZ*-curve) for a freely floating ship, longitudinally balanced at each heel angle. In such cases the *GZ*-curve is ambiguous, as it depends on the way the ship is balanced. Three cases are discussed: when the ship is balanced by rotating her around the trace of water in the midships, around a normal to the ship plane of symmetry (PS), and around a normal to the initial waterplane, fixed to the ship, identical with the curve of minimum stability. In all these cases the direction of the righting moment in space and area under the *GZ*-curves, which is the lowest possible, are preserved. Angular displacements (heel and trim) are the Euler's angles related to the relevant reference axis. The most important features of the *GZ*-curve with free trim are provided. Exemplary calculations illustrate how the way of balancing affects the *GZ*-curves.

NOMENCLATURE

a	height of gravity centre above buoyancy centre in upright position of the ship
B	centre of buoyancy
BP	base plane
BZ	$= -\boldsymbol{r}\cdot\boldsymbol{n}$, height of the centre of gravity above the centre of buoyancy
D	product of inertia of the waterplane
$\boldsymbol{e}$	direction of rotation axis (unit vector normal to plane of rotation)
$\boldsymbol{e}_1$, $\boldsymbol{e}_2$	unit vectors of traces of water in PS and in the midships section
F	centre of floatation (centre of gravity of the waterplane
$\boldsymbol{f}$	unit vector of axis of floatation
H_L	$= R_L - BZ$, longitudinal metacentric height
G	centre of ship gravity
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors of $Oxyz$ system
$\boldsymbol{i}', \boldsymbol{j}, \boldsymbol{k}'$	unit vectors of $Ox'yz'$ system
$\boldsymbol{i}'$	unit vectors of $Ox' = (\cos\theta_0, 0, \sin\theta_0)$
$\boldsymbol{k}'$	unit vectors of $Oz' = (-\sin\theta_0, 0, \cos\theta_0)$
J_1, J_2	principal moment of inertia of the waterplane
J_T	$= J_\xi'' - D''\tan\chi$, transverse moment of inertia
J_η''	longitudinal moment of inertia of the waterplane
L, B, T	length, breadth and mean draught of ship, respectively
l, l_d	righting arm GZ and dynamic arm
l_e	$= \boldsymbol{e}\cdot\boldsymbol{r}$, distance of centre of buoyancy from the plane of rotation
$\boldsymbol{n}$	unit vector normal to waterline, directed upwards
$Oxyz$	coordinate system, fixed to ship
$Ox'yz'$	system $Oxyz$ rotated by angle θ_0 around the axis Oy
$OXYZ$	coordinate system, fixed to the plane of rotation
P	weight of ship
PS	plane of symmetry
$\boldsymbol{r}$	$\equiv \boldsymbol{GB} = (x_B - x_G, y_B - y_G, z_B - z_G)$, radius vector of the centre of buoyancy relative to the ship centre of gravity
r_B	$= J_T/V$, transverse metacentric radius BM
R_L	$= J_\eta''/V$, longitudinal metacentric radius
V	volumetric displacement of ship
$\boldsymbol{w}$	unit vector of the trace of water on initial waterplane
ϑ	angle between traces of water-level and PS in BP
ϑ'	angle between traces of water-level and PS on initial waterplane
Θ	angle of inclination of x-axis relative to sea level
α	angle between BP and water-level
α'	angle between initial waterplane and water-level
β	angle between traces of water in PS and midships
β'	angle of inclination of axis of rotation e with respect to trace of PS on the waterplane
χ	angle between axis of floatation and axis of rotation
Δ	$= V\gamma$, ship buoyancy (weight of displaced water)
ϕ	angle of deviation of PS from the vertical, identical with angle of inclination of y-axis relative to sea level
γ'	angle between principal axis of waterplane and the axis of rotation $\boldsymbol{e}$ from the range $\langle -45°, 45°\rangle$
γ	specific gravity of water
η	rotation angle of plane of rotation in general case
φ	angle of inclination of trace of water in midships relative to y-axis of ship
θ	angle of inclination of trace of water in PS relative to x-axis of ship
θ_0	angle of ship trim at upright position
ρ	density of water
$\xi\eta$	co-ordinate system of the waterplane (ξ-axis coincides with the trace of water in the PS)
$\xi'\eta'$	central system of the waterplane, parallel to system $\xi\eta$
$\xi''\eta''$	central system of the waterplane, where ξ''-axis is parallel to the axis of rotation $\boldsymbol{e}$
$\xi_1\eta_1$	system of principal axes of inertia of waterplane
τ	angle of rotation of the waterplane around axis transverse to axis of rotation $\boldsymbol{e}$
Ψ	$= \psi + \vartheta'$, twist – angle between traces of water-level and PS on initial waterplane for a rig with changed orientation relative to the wind direction
ψ	azimuth – angle between the wind impact plane and PS

1. INTRODUCTION

The *GZ*-curve is the basis for the assessment of ship stability. Until 2008 for intact ships classification societies required the *GZ*-curve to be calculated at level keel. Nowadays, it is commonly calculated for freely floating ships.

For the intact ship it is practically meaningless which mode of calculations is employed: *fixed trim*, constant during heeling, or a *varying trim* as for a freely floating ship, which changes trim depending on longitudinal equilibrium. This is due to minor asymmetry of the ship relative to the midships. However, for damaged ships and platforms the situation is opposite. The mode of calculations is therefore important, as it affects the *GZ*-curve after the immersion of the deck edge in water. The righting arm GZ means here the distance between the lines of action of buoyancy and gravity forces at a given heel angle in still water.

It can be demonstrated that the *GZ*-curve for a free trim is equal to or smaller than that for a fixed trim. For this reason, the *GZ*-curve is nowadays obligatorily calculated for a freely floating ship. In such cases, however, we face the problem of understanding the angle of heel, as it is then an ambiguous notion, manifested in various definitions of this angle and, hence, various *GZ*-curves.

The stability of a freely floating ship is a relatively new issue, explored mainly by Vassalos *et al* (1985), van Santen (1986), the author Pawlowski (1991, 1992a, 1992b, 2005 & 2013), and others.

2. HISTORICAL OUTLINE

Why a body floats in a fluid had been already known in the antiquity since the times of Archimedes (around 287–212 BC). However, how to assess and investigate the stability of floating bodies had not been known until the discovery of the Newtonian laws. In 1746 Bouguer introduced the notion of the metacentre and the metacentric height as a measure of initial stability (Bouguer, 1746). In 1749 Euler delivered the equation for the coefficient of stiffness, and a theorem on the equi-volume waterplanes (Euler, 1749). A thorough survey of the development of the theory of stability of ships since the antiquity is provided in (Nowacki & Ferreiro, 2009). In 1796 Atwood published a method for calculating the righting arm for given heel angle, based on a shifted wedge volume method (Atwood, 1796). From this method it follows that freeboard is crucial for the stability of ships. Nonetheless, for over a hundred years only the initial metacentric height $h_0 \equiv GM$ was used for assessing ship stability. It is stability related accidents at the end of the XIX century that led to a conclusion that the use of the *GM* as the sole criterion is far insufficient for the appraisal of stability, and pointed to the importance of freeboard and the *GZ*-curve.

The metacentric height, which otherwise is an important index of stability, allows neither for direct estimation of the stability range, nor the maximum righting lever. In this context the widely described sinking of HMS *Captain* in 1870 is worth mentioning, with her metacentric height of 0.79 m (White, 1879). The ship capsized during a storm in the Bay of Biscay, whereas the accompanying battleship *Monarch* of a similar size and characteristics, survived the storm unharmed, despite a smaller metacentric height of 0.73 m. The fact was very surprising for the naval architects at that time. It is very easy to explain the accident, if one observes the very different freeboards of the two ships: the *Captain* had a freeboard of 1.98 m, while the *Monarch* of 4.27 m. As a result, despite the smaller metacentric height, the *GZ*-curve of the *Monarch* had much better parameters than that of the *Captain*, whose GZ_{max} = 0.55 m instead of 0.25 m, ϕ_{max} = 40º, instead of 19º, and the range of stability 70º, instead of 54º.

The *Captain's* accident gave evidence that the metacentric height alone is an insufficient measure of stability safety and made it necessary to pay attention to the stability of ships at large angles of heel. As a result, at the end of the XIX century the curve of righting arms (*GZ*-curve) began to be widely used for the assessment of ship stability termed also the *Reed's* curve in memory of their propagator (Reed, 1885). The first *GZ*-based stability criteria appeared as late as in 1939, provided by Rahola, (1939). These are recommendations on minimum values of some parameters related to the *GZ*-curve, extracted from the analyses of the *GZ*-curves for ships that capsized during service and for those regarded as safe. At the end of the 1960s the said criteria were adopted by IMCO (Intergovernmental Maritime Consultative Organisation, established in 1958), presently IMO (International Maritime Organisation since 1982), and they are in force until today, supplemented by the Weather Criterion (IMO, 2009a).

Though the *GZ*-curve had been used for stability assessment of intact ships for more than a century, the stability of damaged ships until recently had been assessed with the metacentric height and freeboard. The previous SOLAS conventions were happy with the residual freeboard as low as three inches and the metacentric height of two inches. With such parameters, the *GZ*-curves are marginal. A change took place as late as in 1990, when the *GZ*-curve was standardised with the help of SOLAS 90 criteria (IMO, 2009b). However, these criteria did not provide a real progress, as they were introduced by purely administrative decision, not supported by any evidence. Hence, they had alleged rather real link to actual safety of ships in damaged condition. A breakthrough took place in 1995 when the mechanism of ship capsizing in damaged condition was revealed (Pawlowski, 1995; Vassalos, Pawlowski & Turan, 1996/97; Pawlowski, 2007a & 2007b). It makes it possible to link the critical sea state and damaged stability at the moment of capsizing applying only static calculations, like for calculating the *GZ*-curves.

3. FORMULATION OF THE PROBLEM

Until recently almost all widely known methods for calculating the *GZ*-curve assumed the ship at level keel. This meant indirectly that the centre of buoyancy *B* was supposed to be free of longitudinal displacements, i.e., when the ship heeled it moved strictly in a frame plane. There was no need for considering earlier a different situation, as the *GZ*-curves were calculated solely for intact ships, for which the foregoing assumption was almost ideally valid. However, in situations when the centre of buoyancy undergoes longitudinal displacements, which takes place when the waterplane is asymmetrical with respect to the plane of rotation, as in the case of ships with low *L*/*B* ratio, or ships in damaged condition, this fact cannot be any longer ignored and the calculations have to be carried out for a freely floating object, longitudinally balanced. Determination of the *GZ*-curve in such cases becomes ambiguous and the problem has to be fine-tuned by determining the way the ship is balanced.

It is worth emphasising that angular rotations of a freely floating object go beyond the basic ship theory. In the classic ship theory the *GZ*-curve is determined for a ship with fixed trim, performing rotations of one degree of freedom. These are elementary rotations, understood by all. Meanwhile, a freely floating ship varies its trim during heeling, that is to say, it performs a rotation of two degrees of freedom. Such a rotation is spatial, much more intricate than that of one degree of freedom. For this reason, and to make the calculations easier vector calculus is applied in this work.

Orientation of a body in space is defined by three Euler's angles, related to a given *reference* axis (Figure 1). In the case of a freely floating ship, two Euler's angles are used, as the third one, describing the azimuth, is irrelevant, as by definition it is constant. One of the two angles plays the role of the angle of heel, while the other – the angle of trim. In the subject literature they are called frequently as generalised heel and trim angles. The Euler's angles are degrees of freedom, i.e. they can be changed independently of each other. A plane normal to the reference axis has no name in mechanics; for convenience we will call it as the *reference plane*. One rotation is around the reference axis, and the other around the *line of nodes NN*, i.e. the trace of water at the plane of reference.

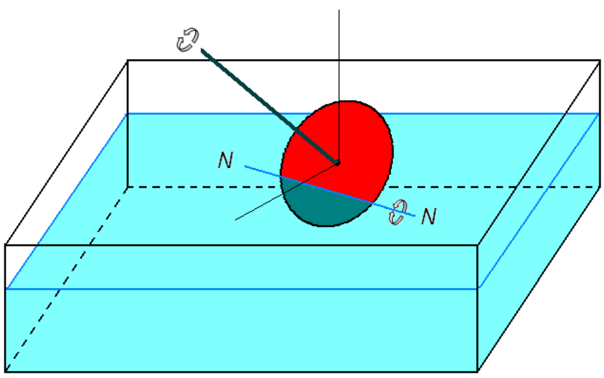

Figure 1. Euler's angles

The reference axis is customarily one of the axes of the coordinate system. There are then three possible reference axes, three reference planes, normal to them, and three lines of nodes. However, a reference axis can be any axis, if necessary.

When a line of nodes is the trace of water in the midships, the Euler's angles are related to the *x*-axis, normal to the midships, denoted by φ and Θ. The first one is the angle of heel, i.e. the angle of inclinations of the trace of water in the midships relative to the *y*-axis, while the other is the trim angle, i.e. the angle of inclination of the *x*-axis relative to a horizontal plane (sea level). The reference plane is any frame plane (station), not necessarily the midships. If the ship is trimmed in an upright position, the Euler's angles are related to the x'-axis, normal to vertical frame planes, denoted by φ' and Θ'. The first one is the angle of inclinations of the trace of water in the vertical frame planes relative to the *y*-axis, while the other is the angle of inclination of the x'-axis with respect to the horizontal. The vertical frames are deviated from regular frames by an angle of initial trim θ_0, and incline together with the ship.

When a line of nodes is the trace of water in the PS, the Euler's angles are related to the *y*-axis, normal to the PS, denoted by ϕ and θ. The first one is the angle of heel, i.e. the angle of rotation of the PS around the trace of water, equal to the angle of inclination of the *y*-axis relative to the horizontal (sea level), while the other is the angle of trim, i.e. the angle of rotation of the PS around the *y*-axis.

When a line of nodes is the trace of water in the initial waterplane (waterplane in an upright position that inclines together with the ship), the Euler's angles are related to the z'-axis, normal to the initial waterplane (when the ship in an upright position at level keel, the reference axis is the *z*-axis, normal to the BP). The Euler's angles are denoted by α' and ϑ' or by α and ϑ, respectively. The first one is the angle of heel, i.e. the angle of rotation of the initial waterplane around the line of nodes, equal to the angle of deviation of the z'-axis from the vertical, while the other one is the angle of trim, termed also as twist, i.e. the angle of rotation of the initial waterplane around the z'-axis, equal to the angle between the traces of water and PS in the initial waterplane. The reference plane is also any plane that is parallel to the initial waterplane. For a ship at level keel this can be in particular the BP.

For a ship heeled with fixed trim, all the three angles of heel are the same, i.e. $\varphi' = \phi = \alpha'$, while the trim angles vanish, i.e. $\Theta' = \theta = \vartheta' = 0$. If a ship is not restrained, then at a given heel angle, she will assume a trim to be longitudinally balanced. In the first case, she will trim (rotate) vertically around the trace of water in the midships (Figure 2), in the second – around the *y*-axis (Figure 3), and in the third case – around the z'-axis (Figure 4). In the last two cases the ship trims in oblique planes.

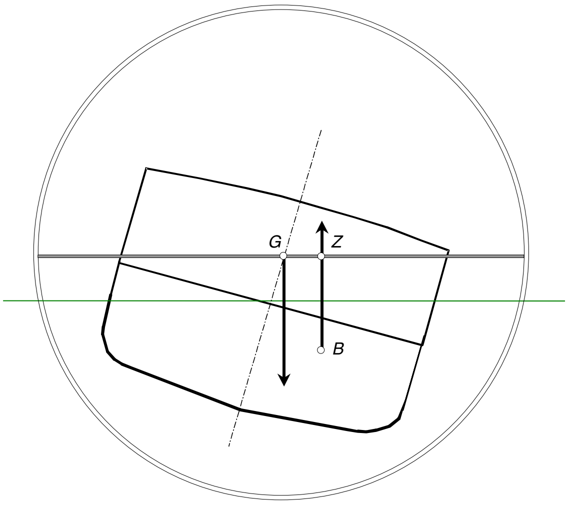

Figure 2. Vertical trimming of the ship

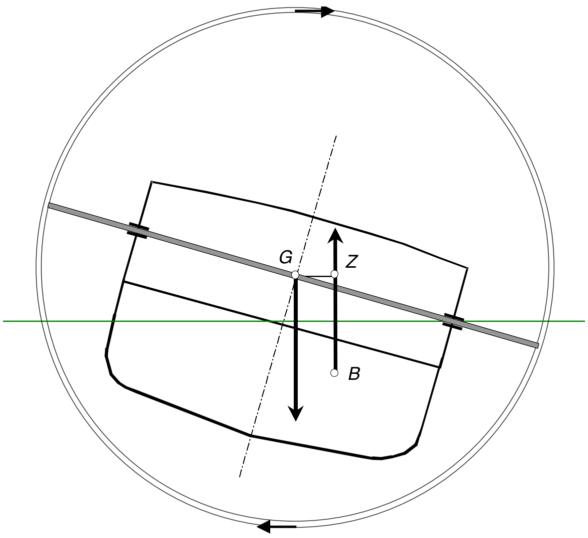

Figure 3. Oblique trimming around the *y*-axis

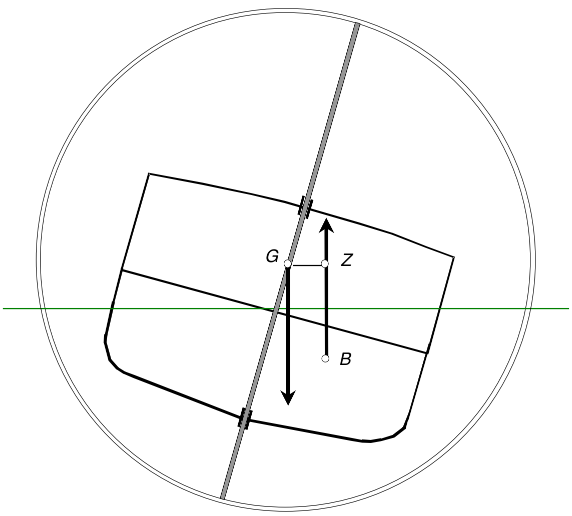

Figure 4. Oblique trimming around the *z*'-axis

Note that for the trim angle Θ' = 90° (when the ship assumes a vertical position) the angle φ' looses the meaning of the heel angle, while for the angle ϕ = 90° the trim angle θ is indeterminate. Only for the reference axis *z*', the trim angle ϑ' is definite, when the heel angle α' = 90°.

Longitudinal balance occurs when the centre of buoyancy is at a vertical plane, termed the *plane of rotation*, passing through the centre of ship gravity. In the first case, the said plane is *parallel* to the line of nodes, while in the two other cases – *perpendicular*. As the line of nodes is fixed in space, the direction of the righting moment is also fixed in space. Hence, the curve of centre of buoyancy is strictly flat in space, lying in the plane of rotation (for a ship with fixed trim, the said curve is a projection of a spatial curve on the plane of rotation). A unit vector, normal to the plane of rotation, termed the *axis of rotation*, denoted further down by ***e***, is also fixed in space.

Calculations of the *GZ*-curve for a freely floating ship are carried out under the following assumptions:

- *The ship is inclined by a pure heeling moment, acting statically*. It means that ship inclinations are equi-volume
- *The vector of the heeling moment is strictly horizontal*. Otherwise, the heeling moment would have a vertical component that would rotate the ship around a vertical axis
- *The vector of the heeling moment is normal to the plane of rotation*. Otherwise, the ship would not be longitudinally balanced
- *At each heel angle the ship is in static equilibrium*, i.e. the sum of forces and moments acting on her vanish. Hence the ship's weight is equal to her buoyancy, i.e. $P = D$, and the static heeling moment is balanced by the righting moment of the opposite direction
- *The righting moment is formed by a couple of forces*: i.e. the gravity force applied in the ship's centre of gravity and the buoyancy force passing through the ship's centre of buoyancy. These forces are equal to each other and of opposite direction to each other. The moment vector is horizontal and normal to the plane of rotation.

The above assumptions yield some consequences:

- For inclinations with fixed trim the centre of buoyancy need not be in the plane of rotation, therefore the moment acting on the ship has no constant direction in a horizontal plane
- The righting lever $l \equiv GZ$ is the arm of the couple forming the righting moment, measured in the plane of rotation; the said arm is a function of the angle of rotation η of the plane of rotation around the axis of rotation ***e***. The angle of rotation depends on the reference axis. In the second case $\eta \equiv \phi$, in the third case $\eta \equiv \alpha'$. In the first case $\eta = \int d\varphi \cos\Theta$. For the reference axis *Ox*', the angle of rotation has no simple interpretation. This comes from the fact that the reference axis is *not* normal to the plane of rotation (ver-

tical frame). Therefore, $\eta < \varphi$. For ships the angles η and φ are practically the same, since the trim angles Θ are less than 1°. In the case of platforms, the differences between the two angles can be large, due to large trims
- The orientation of the ship to the plane of rotation is ambiguous, as it depends on the adopted line of nodes and related method of balancing; therefore the *GZ*-curves are also ambiguous. The trace of water in the PS (Figure 3) is appropriate for intact ships, as it idealises the direction of the wind heeling moment. On the other hand, the edge of intersection of the initial waterplane with the waterplane is appropriate for damaged ships, where the heeling moment is created by gravitational forces, assuming minimum potential energy at the position of equilibrium
- In the case of objects arbitrarily orientated to wind direction (e.g. semi-submersible units) the PS should be replaced by a *wind impact screen*, perpendicular to the wind direction at an initial position and rotating together with the object. For a ship asymmetrically flooded this is a plane parallel to the principal axis of the waterplane in an upright position. The Euler's angles are related to the system *Ox'y'z'*, fixed to the windscreen
- It can be seen that the projection of the *y*-axis on the horizontal plane is perpendicular to the trace of water in the PS. Hence, this line of nodes strictly corresponds to the direction of the heeling moment due to a shift of cargo in the ship's transverse plane. It applies also to the heeling moment of ro-ro vessels in damaged condition, resulting from the accumulation of water on the car deck when a symmetrical compartment has been flooded in the midships. For the same reason the *GZ*-curve measured by means of the Di Belli method is strictly consistent with the above model of inclinations. In this method, a heel angle of ship model is measured, induced by shifting a weight along an arm perpendicular to the PS, identical with the inclination of the arm relative to the horizontal
- As the righting moment is all the time perpendicular to the plane of rotation, work done by the righting moment is the integral of the moment with respect to the angle of rotation of the plane of rotation η, identical with the heel angle, dependent on the line of nodes. At the same time, this is the least work which is to be performed in order to heel the ship up to a given heel angle. In other words, for the ship with fixed trim or not fully balanced, work of the righting moment is larger.

It is worth emphasising that in space there is only *one* rotation plane (large circle in the said three figures). However, the ship sets differently with respect to it depending on the way of balancing. In the case of the reference axis *x*, longitudinal balance of the ship is achieved by vertical trimming around the trace of water in the frame planes (Figure 2), in the case of the *y*-axis – around a normal to the PS (Figure 3), i.e. around the *y*-axis, and in the case of the axis *z'* – around a normal to the initial waterplane (Figure 4). Hence, the ship after balancing has various orientations relative to the plane of rotation, producing different righting arms, dependent on the way the ship is balanced (the choice of the reference axis). Nonetheless, the direction of the righting moment in space is the same.

4. STABILITY CHARACTERISTICS

A number of stability characteristics, of basic importance for a freely floating ship will be discussed here, such as the angle of rotation, righting arm, moments of inertia of the waterplane (understood as a cross-section of the ship hull by a flat surface of the sea), metacentric radii, axis of floatation, and cross curves of stability. We will start by a description of the waterplane for arbitrarily inclined ship, which is independent of the choice of the reference axis.

4.1 BASIC RELATIONSHIPS

A right hand-side co-ordinate system *Oxyz*, shown in Figure 5, fixed to the ship, is assumed. The *x*-axis is directed forward, the *y*-axis – portside, and the *z*-axis – upwards. An arbitrarily inclined waterplane is described by the equation:

$$z = T_0 + x\tan\theta + y\tan\varphi \qquad (1)$$

in which three independent parameters appear: the angle of inclination of the trace of water θ in the PS relative to the *x*-axis, the angle of inclination of the trace of water φ in the midships relative to the *y*-axis, and the draught T_0 of the *z*-axis. The two angles φ and θ are termed the *analytical angles*. They are positive if a positive increment of x or y corresponds to a positive increment of z. Hence, the trim angle $\theta > 0$ is positive, when the ship is trimmed by bow, while the angle $\varphi > 0$ is positive, when the ship is heeled portside (in Figure 2–4 the ship is inclined to starboard, therefore heel angles are negative in these figures). Both angles are easy to measure, as $\tan\theta = t/L_{pp}$, and $\tan\varphi = \Delta T_{LR}/B$, where $t \equiv \Delta T_{BS}$ is a trim, i.e. the difference of draughts at the bow and stern perpendiculars, and ΔT_{LR} is the difference of draughts at portside and starboard in the midships section.

It is known from analytical geometry that a vector normal to the waterplane, as given by equation (1), is as follows:

$$\boldsymbol{R} = (\tan\theta, \tan\varphi, -1)$$

which is directed downwards, and whose absolute value $R = (1 + \tan^2\theta + \tan^2\varphi)^{1/2}$. Hence, a unit vector, normal to the waterplane, and directed upwards, equals $\boldsymbol{n} = -\boldsymbol{R}/R$. Note that $\boldsymbol{n}$ is defined by the two analytical angles. As the two angles define any other angles, particularly the Euler's angles the unit vector $\boldsymbol{n}$ in terms of the analytical angles is valid for any reference axis.

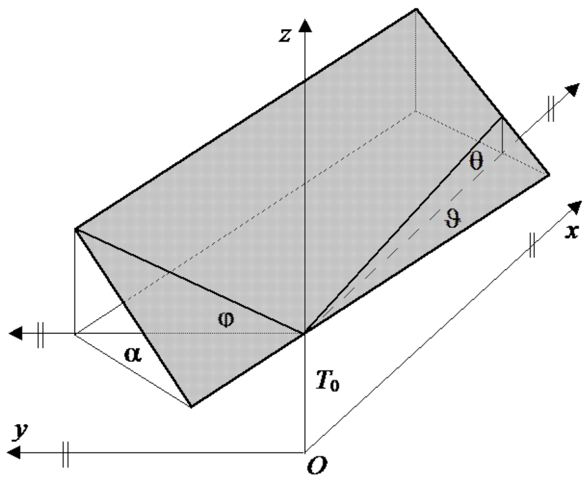

Figure 5. Analytical angles of inclined waterplane

For example, two Euler's angles α and ϑ, and a line of nodes related to the z-axis, are shown in Figure 5. The former is the heel angle and the latter – twist angle.

We will find now some angles of interest for given analytical heel angles. An angle between planes is the same as between vectors normal to them. Hence, the angle α between the waterplane and BP, or an upright waterplane, is given by the equation: $\cos\alpha = \boldsymbol{k}\cdot\boldsymbol{n} = -\boldsymbol{k}\cdot\boldsymbol{R}/R = -R_z/R$. Therefore, $\cos\alpha = 1/R = 1/(1 + \tan^2\theta + \tan^2\varphi)^{1/2}$. Hence,

$$\tan\alpha = (\tan^2\theta + \tan^2\varphi)^{1/2}$$

The sign of the angle α is the same as that of the angle φ. Taking into account that $1/R = \cos\alpha$, components of the unit vector $\boldsymbol{n} = -\boldsymbol{R}/R = -\cos\alpha\boldsymbol{R}$ are as follows:

$$\boldsymbol{n} = (-\tan\theta\cos\alpha,\ -\tan\varphi\cos\alpha,\ \cos\alpha) \tag{2}$$

The trim angle related to the axis Ox, i.e. the angle Θ, is equal to the angle of inclination of the x-axis relative to the sea surface. Hence, $\cos(90° + \Theta) = \boldsymbol{i}\cdot\boldsymbol{n} = n_x$, which is equivalent to: $\sin\Theta = -n_x$. Thus,

$$\begin{aligned}\sin\Theta &= \tan\theta\cos\alpha,\\ \tan\Theta &= \tan\theta\cos\varphi.\end{aligned} \tag{3}$$

The angle of heel related to the trace of water in the PS, denoted by ϕ, is equal to the angle of inclination of the y-axis relative to the surface of the sea. Hence, $\cos(90° + \phi) = \boldsymbol{j}\cdot\boldsymbol{n} = n_y$, which is equivalent to: $\sin\phi = -n_y$. Thus,

$$\begin{aligned}\sin\phi &= \tan\varphi\cos\alpha,\\ \tan\phi &= \tan\varphi\cos\theta.\end{aligned} \tag{4}$$

Note that the angle $\phi \le \alpha$, which follows immediately from the identity $\cos\phi = \cos\alpha/\cos\theta$, obtained by dividing $\sin\phi$ by $\tan\phi$. From equation (4) it follows moreover that the angle $\phi \le \varphi$. Hence, the heel angle ϕ is never greater than the angle α, or the angle φ. However, bearing in mind that for conventional ships the vertical trim angle Θ is below 1°, even for the largest trims, differences between heel angles ϕ, φ and α are imperceptible.

The angle of inclination of the trace of water in the BP relative to the x-axis, denoted by ϑ, is the slope (gradient) of the line in a plane $z = const$. From equation (1) we get immediately that

$$\tan\vartheta = -\tan\theta/\tan\varphi \tag{5}$$

In an upright position, for $\varphi = 0$, equation (5) is indeterminate. In such a case, $\vartheta = 0$. Equivalent forms of equation (5) are as follows:

$$\begin{aligned}\sin\vartheta &= -\tan\theta/\tan\alpha,\\ \cos\vartheta &= \tan\varphi/\tan\alpha.\end{aligned}$$

It is worth noting that traces of water in the PS and midships (or any frame plane), shown in Figure 5, are not generally perpendicular one to another. The angle between them can be easily found with the help of the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of both traces; they look at the directions of the x- and y-axes. Denoting the angle between the said traces by β, then $\cos\beta = \boldsymbol{e}_1\cdot\boldsymbol{e}_2$, where the unit vector of the trace of water in the PS $\boldsymbol{e}_1 = (\cos\theta, 0, \sin\theta)$, while the unit vector of traces of water in the frame planes $\boldsymbol{e}_2 = (0, \cos\varphi, \sin\varphi)$. Hence,

$$\cos\beta = \sin\theta\sin\varphi \tag{6}$$

When both analytical angles are of the same sign, the angle between the unit vectors is acute (which is also seen in Figure 5. Otherwise, the angle is obtuse.

With the help of the above identities, the unit normal vector $\boldsymbol{n}$, given by equation (2), can be easily expressed by the Euler's angles, appropriate for a given reference axis. Eliminating the analytical angles, the following is obtained:

$$\begin{aligned}\boldsymbol{n} &= (-\sin\Theta,\ -\cos\Theta\sin\varphi,\ \cos\Theta\cos\varphi),\\ &= (-\sin\theta\cos\phi,\ -\sin\phi,\ \cos\theta\cos\phi),\\ &= (\sin\vartheta\sin\alpha,\ -\cos\vartheta\sin\alpha,\ \cos\alpha).\end{aligned} \tag{7}$$

The first expression is in terms of the Euler's angles for the x-axis, the second and third – for the y- and z-axis.

4.1 (a) Effect of the initial trim

If the ship has the initial trim θ_0 in an upright position, the Euler's angles are related to the co-ordinate system $Ox'yz'$, as in Figure 6. The axis Ox' is horizontal, i.e. parallel to the sea level, while the axis Oz' is vertical, i.e. normal to the sea level. The initial trim does not change the axis Oy. Hence, it does not change the Euler's angles, related to this axis, while it changes them for the two other axes. As previously, we want to express them in terms of the analytical angles.

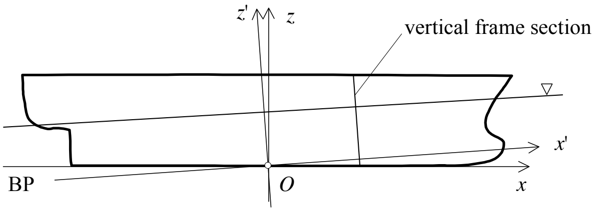

Figure 6. Co-ordinate system for a trimmed vessel

The reference plane for the axis Ox' is a vertical frame section, fixed to the ship, deviated from the regular frame planes by the initial trim angle θ_0 (Figure 6); the angle $\theta_0 > 0$ is positive for bow trim. The trim angle Θ', related to the axis Ox', is equal to the angle of inclination of the axis Ox' relative to the horizontal. Hence, $\cos(90° + \Theta') = \boldsymbol{i}' \cdot \boldsymbol{n}$, where $\boldsymbol{i}' = (\cos\theta_0, 0, \sin\theta_0)$ is a unit vector of the axis Ox'. Hence, $\sin\Theta' = -\boldsymbol{i}' \cdot \boldsymbol{n}$, which yields:

$$\sin\Theta' = (\tan\theta\cos\theta_0 - \sin\theta_0)\cos\alpha$$

If $\theta_0 = 0$, the above equation reduces to equation (3).

The heel angle φ' is equal to the angle between the traces of water and initial waterplane at the reference plane (vertical frame section). The unit vector of the trace of water $\boldsymbol{e}_2'$ at the vertical frame section equals:

$$\boldsymbol{e}_2' = \boldsymbol{n} \times \boldsymbol{i}'/\sin(90° + \Theta') = \boldsymbol{n} \times \boldsymbol{i}'/\cos\Theta' \tag{8}$$

while the other unit vector is identical with the unit vector $\boldsymbol{j}$ of the axis Oy. Therefore, $\cos\varphi' = \boldsymbol{j} \cdot \boldsymbol{e}_2'$. Hence,

$$\cos\varphi' = (\cos\theta_0 + \sin\theta_0\tan\theta)\cos\alpha/\cos\Theta'$$

When $\theta_0 \to 0$, $\varphi' \to \varphi$, since $\cos\varphi' = \cos\alpha/\cos\Theta$. Substituting for $\cos\Theta = \cos\alpha/\cos\varphi$, in the limit we get $\cos\varphi' = \cos\varphi$, which implies $\varphi' = \varphi$.

The reference plane for the axis Oz' is any plane parallel to the waterplane in an upright position, fixed to the ship. Its unit normal vector $\boldsymbol{k}' = (-\sin\theta_0, 0, \cos\theta_0)$ is identical with a unit vector of the axis Oz'. The heel angle α' is given by the equation: $\cos\alpha' = \boldsymbol{k}' \cdot \boldsymbol{n}$, which yields:

$$\cos\alpha' = (1 + \tan\theta_0\tan\theta)\cos\theta_0\cos\alpha$$

The trim angle ϑ' (twist angle) is the angle between the traces of water and PS in the reference plane (initial waterplane). The unit vectors of these traces are as follows: $\boldsymbol{w} = \boldsymbol{k}' \times \boldsymbol{n}/\sin\alpha'$ and $\boldsymbol{i}'$. Hence, the twist angle is given by: $\cos\vartheta' = \boldsymbol{i}' \cdot \boldsymbol{w} = \boldsymbol{i}' \cdot (\boldsymbol{k}' \times \boldsymbol{n})/\sin\alpha' = -n_y/\sin\alpha'$. Thus:

$$\cos\vartheta' = \tan\varphi\cos\alpha/\sin\alpha'$$

The sign of the angle ϑ' is opposite to the sign of the angle θ, which follows from equation (5), i.e. it is negative, when the trim is on the bow. If $\theta_0 = 0$, then $\alpha' = \alpha$, while $\vartheta' = \vartheta$, which can be easily shown. A change of the trim angle does not affect the heel angle, which is not seen at the first glance. And this holds for any reference axis.

4.1 (b) Wind impact screen

Consider now the angles related to the wind impact plane, deviated from the PS by an angle ψ, termed the *azimuth*, wherein $\psi > 0$, if it is anti-clockwise. A system $Ox''y'z'$ is fixed to this plane, rotated by the angle ψ around the axis Oz' relative to the system $Ox'yz'$. By definition, the said plane is perpendicular to the direction of the wind. When $\psi = 0$, the impact plane coincides with the PS.

The unit vectors $\boldsymbol{i}''$ and $\boldsymbol{j}'$ of the system $Ox''y'z'$ are rotated by the angle ψ relative to the unit vectors $\boldsymbol{i}'$ and $\boldsymbol{j}$. Hence, taking their projections on the system axes, we get:

$$\begin{aligned} \boldsymbol{i}'' &= \boldsymbol{i}'\cos\psi + \boldsymbol{j}\sin\psi \\ &= (\cos\theta_0\cos\psi, \sin\psi, \sin\theta_0\cos\psi), \\ \boldsymbol{j}' &= -\boldsymbol{i}'\sin\psi + \boldsymbol{j}\cos\psi \\ &= (-\cos\theta_0\sin\psi, \cos\psi, -\sin\theta_0\sin\psi). \end{aligned} \tag{9}$$

In the case of the reference axis Oy', normal to the wind impact plane, playing a role of the reference plane, the line of nodes is the trace of water in the said plane $\boldsymbol{e}_1'$. This trace is at the same time the axis of rotation related to the reference axis Oy'. The unit vector $\boldsymbol{e}_1'$ results from trimming of the ship by an angle θ' relative to the axis Ox''. In other words, rotating the unit vectors $\boldsymbol{i}''$ and $\boldsymbol{k}'$ by the angle θ' around the axis Oy' the unit vector $\boldsymbol{i}''$ becomes the unit vector $\boldsymbol{e}_1'$, and $\boldsymbol{k}'$ becomes $\boldsymbol{k}''$. Hence,

$$\begin{aligned} \boldsymbol{e}_1' &= \boldsymbol{i}''\cos\theta' + \boldsymbol{k}'\sin\theta', \\ \boldsymbol{k}'' &= \boldsymbol{k}'\cos\theta' - \boldsymbol{i}''\sin\theta'. \end{aligned} \tag{10}$$

Finally, the unit vector $\boldsymbol{n}$ results from the rotation of the ship (waterplane) around the trace of water in the wind impact plane $\boldsymbol{e}_1'$ by an angle of heel ϕ', i.e. the angle of inclination of the y'-axis relative to the horizontal. Hence,

$$\boldsymbol{n} = \boldsymbol{k}''\cos\phi' - \boldsymbol{j}'\sin\phi' \tag{11}$$

The angles θ' and ϕ' are the Euler angles, related to the reference axis Oy'. The former results from longitudinal balancing of the ship. The knowledge of the unit vector $\boldsymbol{n}$ defines the analytical angles, essential for calculating the geometric characteristics of the waterplane and ship's hull. When $\psi = \theta_0 = 0$, equation (11) reduces to the second expression in equation (7).

For the reference axis Oz', the line of nodes is a *given* trace of water in the initial waterplane, playing the role of the reference plane; the unit vector of this trace is denoted by $\boldsymbol{w}$. In an upright position, $\boldsymbol{w} = \boldsymbol{i}'$. It is at the same time the axis of rotation $\boldsymbol{e}$, related to this axis of reference. Obviously, $\boldsymbol{w} = \boldsymbol{k}' \times \boldsymbol{n}/\sin\alpha'$. However, this equation cannot be used now, as the unit vector $\boldsymbol{n}$ is treated here as given, while the unit vector $\boldsymbol{w}$ is resultant, whereas it should by the other way round.

Both unit vectors $\boldsymbol{w}$ and $\boldsymbol{n}$ result from rotations. The unit vector $\boldsymbol{n}$ results from the rotation of the ship (waterplane) around the trace of water $\boldsymbol{w}$ on the initial waterplane by a heel angle α', whereas the unit vector $\boldsymbol{w}$ of the trace of water on the initial waterplane results from the rotation of $\boldsymbol{w}$ around the unit vector $\boldsymbol{k}'$ by a trim (twist) angle ϑ' (Figure 1). They are given by equations for the rotation of a vector by a given angle in an appropriate base of unit vectors:

$$\boldsymbol{n} = \boldsymbol{k}'\cos\alpha' + (\boldsymbol{w}\times\boldsymbol{k}')\sin\alpha', \qquad (12)$$
$$\boldsymbol{w} = \boldsymbol{i}''\cos\vartheta' + \boldsymbol{j}'\sin\vartheta',$$

where the unit vectors $\boldsymbol{i}''$ and $\boldsymbol{j}'$ are given by equations (9), α' is the angle of heel, i.e. the angle of inclination of the initial waterplane relative to the horizontal, and ϑ' is the trim angle measured in the initial waterplane from the direction $\boldsymbol{i}''$ (when $\vartheta' > 0$, the twist is by aft); these are the Euler angles, related to the axis *Oz*'. The angle ϑ' results from longitudinal balancing of the ship. The knowledge of the unit vector $\boldsymbol{n}$ defines the analytical angles, essential for calculating the geometric characteristics of the waterplane and ship's hull.

The unit vector $\boldsymbol{w}$ is rotated in relation to the unit vector $\boldsymbol{i}'$ by the angle $\Psi = \psi + \vartheta'$, equal to the sum of the azimuth and the angle of trim (twist). Hence, both unit vectors in equation (12) can be expressed simpler in the base of the system *Ox*'*yz*':

$$\boldsymbol{n} = \boldsymbol{i}'\sin\alpha'\sin\Psi - \boldsymbol{j}\sin\alpha'\cos\Psi + \boldsymbol{k}'\cos\alpha', \qquad (13)$$
$$\boldsymbol{w} = \boldsymbol{i}'\cos\Psi + \boldsymbol{j}\sin\Psi.$$

In view of the fact that the rotation of the unit vector $\boldsymbol{w}$ by an angle Ψ relative to $\boldsymbol{i}'$ can take place in a *horizontal* initial waterplane *before* heeling, van Santen calls this rotation as "twist" (Santen van, 2009 & 2019), without a clear indication that this is one of the two Euler's angles, related to trim, measured in the initial waterplane *after* heeling (Figure 4).

The following identities result from equations (12) and (13): $\boldsymbol{k}'\cdot\boldsymbol{n} = \cos\alpha'$, $\boldsymbol{k}'\times\boldsymbol{n} = \boldsymbol{w}\sin\alpha'$, $\boldsymbol{i}'\cdot\boldsymbol{w} = \cos\Psi$, $\boldsymbol{i}'\times\boldsymbol{w} = \boldsymbol{k}'\sin\Psi$. When $\psi = 0$, $\Psi = \vartheta'$. For a trimmed ship in an upright position equations (12) yield:

$$\boldsymbol{w} = (\cos\vartheta',\ \sin\vartheta',\ 0), \qquad (14)$$
$$\boldsymbol{n} = (\sin\vartheta'\sin\alpha',\ -\cos\vartheta'\sin\alpha',\ \cos\alpha').$$

For a ship at level keel, the angles α' and ϑ' are replaced by α and ϑ. The unit vector $\boldsymbol{n}$ becomes then identical with the third expression in equation (7).

In the case of the reference axis *x*'', it is easier to find the final position of the object heeling it first by an angle φ' around the axis *Ox*'', described by the unit vector $\boldsymbol{i}''$, and next trimming it by an angle Θ' around the trace of water in a plane normal to the axis *Ox*'', described by the unit vector $\boldsymbol{e}_2'$. As a result of the first rotation around the axis *Ox*'' new unit vectors $\boldsymbol{e}_2'$ and $\boldsymbol{k}''$ are obtained, while the second rotation around $\boldsymbol{e}_2'$ yields the unit vector $\boldsymbol{n}$: Hence:

$$\boldsymbol{e}_2' = \boldsymbol{j}'\cos\varphi' + \boldsymbol{k}'\sin\varphi', \qquad (15)$$
$$\boldsymbol{k}'' = \boldsymbol{k}'\cos\varphi' - \boldsymbol{j}'\sin\varphi',$$
$$\boldsymbol{n} = \boldsymbol{k}''\cos\Theta' - \boldsymbol{i}''\sin\Theta',$$

where the unit vectors $\boldsymbol{i}''$ and $\boldsymbol{j}'$ are given by equations (9). The angles φ' and Θ' are the Euler's angles, related to the reference axis *Ox*'; the latter results from longitudinal balancing of the ship.

A change of orientation of the object in the horizontal introduces a third Euler angle – the azimuth ψ. However, it follows from equations (13) that at least for the axis *Oz*' the unit vector $\boldsymbol{n}$, describing the attitude of the ship relative to the horizontal, depends on two Euler's angles: the heel angle α' and twist $\Psi = \psi + \vartheta'$. For other reference axes things are more complicated – the unit vector $\boldsymbol{n}$ depends on three Euler's angles, not on two. It means that in such cases the relationship between the two Euler's angles (heel and trim) and analytical angles φ and θ is affected by the azimuth ψ.

4.2 RIGHTING ARM

To calculate the *GZ*-curve, we have to choose a reference axis. Generally, there are three possibilities. In the case of the reference axis *Ox*', commonly used for calculating the *GZ*-curves with free trim, e.g. in the Napa system, Proteus, Stataw, WinSEA, and in many other computer programs, the plane of rotation is a vertical frame station, parallel to the trace of water in the frames (Figure 2); in the case of the axis *Oy*, it is normal to the trace of water in the PS (Figure 3), and in the case of the axis *Oz*' – normal to the trace of water in the initial waterplane (Figure 4).

The plane of rotation at which the ship is balanced is defined by a unit vector $\boldsymbol{e}$, normal or parallel to the line of nodes, depending on the reference axis. When the line of nodes is the trace of water in the midships (Figure 2), the axis of rotation $\boldsymbol{e} = \boldsymbol{e}_2\times\boldsymbol{n}$, where $\boldsymbol{e}_2 = (0, \cos\varphi, \sin\varphi)$ is the unit vector of the trace of water in the midships. Substituting for $\boldsymbol{n}$ equation (7), the axis of rotation is:

$$\boldsymbol{e} = (\cos\Theta,\ -\sin\Theta\sin\varphi,\ \sin\Theta\cos\varphi) \qquad (16)$$

When the ship has an initial trim, the unit vector $\boldsymbol{e}_2$ is replaced by the vector $\boldsymbol{e}_2'$, given by equation (8), and when the azimuth $\psi \neq 0$, the unit vector $\boldsymbol{e}_2$ is replaced by $\boldsymbol{e}_2'$, given by equation (15). When the line of nodes is the trace of water in the PS (Figure 3), $\boldsymbol{e} = \boldsymbol{e}_1$, where $\boldsymbol{e}_1 = (\cos\theta, 0, \sin\theta)$ is a unit vector of the trace of water in the PS, and when the line of nodes is the trace of water in the wind impact plane, the rotation axis $\boldsymbol{e} = \boldsymbol{e}_1'$, where $\boldsymbol{e}_1'$ is given by equation (10). When the line of nodes is the trace of water in the initial waterplane $\boldsymbol{w}$ (Figure 4), the rotation axis $\boldsymbol{e} = \boldsymbol{w}$, where the unit vector $\boldsymbol{w}$ is given by equation (12), valid both for the ship at level keel, trimmed at an upright position, or rotated by a certain azimuth ψ.

The three axes of rotation diverge, if trim varies in the course of inclinations. For example, the axis of rotation $\boldsymbol{e}$, related to the reference axis *Ox*, is deviated from the trace of water in the PS by an angle $\gamma_1 = \beta - 90°$, where β is the angle between the traces of water in the midships and PS, given by equation (6). Further, the axis of rotation $\boldsymbol{e} = \boldsymbol{k}\times\boldsymbol{n}/\sin\alpha$, related to the reference axis Oz, is deviated from the trace of water in the PS by an angle γ_3, which can be found from the equation: $\boldsymbol{e}_1\times\boldsymbol{e} = \sin\gamma_3\,\boldsymbol{n}$. Hence, $\sin\gamma_3 = -\sin\theta/\sin\alpha$. As can be seen, the axes of rotation coincide with each other, when there is no trim.

The plane of rotation rotates around the axis of rotation $\boldsymbol{e}$, whereas the waterplane, i.e. the ship, rotates around an instantaneous *axis of floatation* $\boldsymbol{f}$, oblique relative to the axis of rotation. The axis of floatation $\boldsymbol{f}$ is understood as the edge of intersection of two waterplanes inclined relative to one another at an infinitely small angle. In the case of equi-volume waterplanes it passes through the centre of floatation F, i.e. the centre of gravity of the waterplane. The above follows from the Pappus–Guldinus' theorem, known in ship theory as the Euler's theorem on equi-volume waterplanes. This theorem says nothing about orientation of the axis of floatation, defined by a unit vector $\boldsymbol{f}$, discussed below. In mechanics, the axis of floatation is termed the *instantaneous axis of rotation*. To find the axis of floatation it is necessary to know moments of inertia of the waterplane, which is not trivial in the case of a freely floating ship.

When the ship is being inclined the displacement remains constant, whereas the centre of buoyancy B moves in the plane of rotation (large circle in Figure 2–4), normal to the axis of rotation $\boldsymbol{e}$. Hence, it has to satisfy the equation of the plane of rotation: $\boldsymbol{e}\cdot\boldsymbol{r} = 0$, where $\boldsymbol{r} \equiv \boldsymbol{GB} = (x_B - x_G, y_B - y_G, z_B - z_G)$ is the radius vector of the centre of buoyancy relative to the ship centre of gravity. The quantity $\boldsymbol{e}\cdot\boldsymbol{r} \equiv l_e$ is a longitudinal component of the righting arm, identical with a distance of the centre of buoyancy from the plane of rotation (if $l_e > 0$, it is forward of the plane of rotation). For given volume displacement V and heel angle (angle of rotation of the plane of rotation η) the longitudinal component $l_e = \boldsymbol{e}\cdot\boldsymbol{r}$ is a function of the trim angle.

The righting moment is given by the equation $\boldsymbol{M} = \boldsymbol{r}\times\boldsymbol{n}\Delta$, where $\Delta = \gamma V$ is the ship buoyancy. Vector $\boldsymbol{M}$ is parallel to the rotation axis $\boldsymbol{e}$, hence: $M = \boldsymbol{e}\cdot(\boldsymbol{r}\times\boldsymbol{n})\Delta$. The righting arm $GZ = M/\Delta$ is therefore given by the equation:

$$GZ = \boldsymbol{e}\cdot(\boldsymbol{r}\times\boldsymbol{n}) \tag{17}$$

The effect of the plane of rotation (reference axis) on the GZ-curve can be clearly seen in equation (17), where the righting lever $l \equiv GZ$ depends on $\boldsymbol{r}$, $\boldsymbol{n}$ and $\boldsymbol{e}$. For the same analytical angles φ and θ, the vector $\boldsymbol{r}\times\boldsymbol{n}$ is the same but the different rotation planes have different $\boldsymbol{e}$, which in turn gives different righting levers l. Nonetheless, the areas under GZ-curves for various reference axes have to be the same. The proof is simple. If the large circles in Figure 2–4 is rotated so that the righting arm $GZ = 0$ vanishes, then the maximum work is performed, i.e., the ship reaches maximum of potential energy. Since, only one maximum exists, it has to be independent of the choice of the reference axis. That is to say, the areas under GZ-curves are conserved. Hence, if ranges of the GZ-curves for various reference axes are different, as it happens in the case of rigs, then in the descending part of these curves they have to intersect with each other. However, the differences between them are modest.

The differences between the various GZ-curves are solely attributed to the axes of rotation $\boldsymbol{e}$. Deviations of these axes from the trace of water in the PS are described by the angles γ_1, γ_2 and γ_3 (normally, $\gamma_2 = 0$). For the GZ-curve of minimum stability $\gamma_1 = \gamma_2 = \gamma_3$, which means a common plane of rotation, independent of the reference axis. If the axes $\boldsymbol{e}$ were the same for the various reference axes, GZ-curves would be the same. The larger angles between the rotation axes, the larger differences between the GZ-curves. Typically, the axes of rotation $\boldsymbol{e}$ for the reference axes x' and y are nearly parallel, therefore the GZ-curves for the two axes are almost identical. However, they differ somewhat from the GZ-curve for the axis z'.

As can be seen, the basis for finding the GZ-curve with free trim is the knowledge of co-ordinates of the centre of buoyancy B, the rotation axis $\boldsymbol{e}$, and the normal $\boldsymbol{n}$ to the waterplane. The result of calculations is a curve of righting arms with the lowest values, called the *GZ-curve of minimum stability*, introduced by Siemionov-Tiań-Szański (1960).

4.3 PROPERTIES OF THE GZ-CURVE

During equi-volume inclinations the centre of buoyancy B moves in the plane of rotation along a flat curve termed the *curve of centres of buoyancy* (Figure 7). The curvature of this curve is termed the *metacentric radius*, denoted by $r_B \equiv BM$, derived later. The metacentric height $h \equiv ZM$ is equal to:

$$h = {}^{d}\!/_{d\eta}\, l = r_B - BZ \tag{18}$$

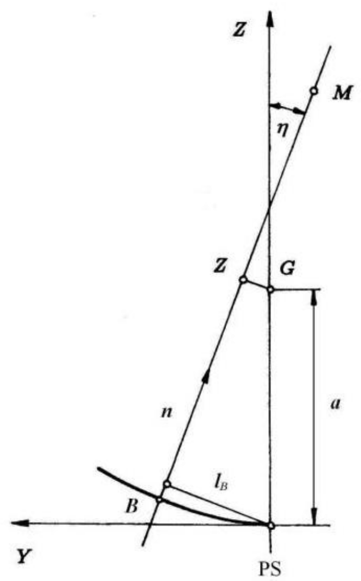

Figure 7. Rotation plane

where $r_B \equiv BM$ is the metacentric radius, $BZ = -\boldsymbol{r}\cdot\boldsymbol{n}$ is the height of ship centre of gravity above its centre of buoyancy (Figure 2–4), $\boldsymbol{r} = \boldsymbol{GB}$ is the radius-vector of ship centre of buoyancy relative to its centre of gravity. Equation (18) can be immediately obtained by considering the line of action of buoyancy force in the plane of rotation (Figure 7) for heel angle increased by $d\eta$, where the angle of rotation $\eta = \phi$ or α, depending on the line of nodes (the system B_0YZ is fixed to the plane of rotation, whose origin is at an initial position of the centre of buoyancy B_0). The metacentric height can be also obtained by differentiating the righting arm $l \equiv GZ$, given by equation (17), with respect to heel angle (angle of rotation) in the ship-fixed reference system. This derivative is given by the equation:

$$GZ' = \boldsymbol{e}'\cdot(\boldsymbol{r}\times\boldsymbol{n}) + \boldsymbol{e}\cdot(\boldsymbol{r}'\times\boldsymbol{n}) + \boldsymbol{e}\cdot(\boldsymbol{r}\times\boldsymbol{n}') \\ = r_B + \boldsymbol{r}\cdot\boldsymbol{n}$$

identical with equation (18), where ' stands for differentiating respective to the heel angle η. It can be shown that the first term $\boldsymbol{e}'\cdot(\boldsymbol{r}\times\boldsymbol{n})$ vanishes (it is sufficient to observe that the three vectors are coplanar, i.e. lie in the plane of rotation), the second one is the metacentric radius $r_B = BM$, and the third one equals $\boldsymbol{r}\cdot\boldsymbol{n}$.

Work done by the righting moment M is given by:

$$L = \int_0^\eta M d\eta = \Delta\int_0^\eta l\, d\eta = \Delta\, l_d, \qquad (19)$$
$$l_d = \int_0^\eta l\, d\eta,$$

where Δ is buoyancy of the ship, and l_d is the *dynamic arm*, the same as the first integral curve of the *GZ*-curve, i.e. the area under the *GZ*-curve. Considering rotation of the plane of rotation by an angle $d\eta$ (Figure 7), one can easily demonstrate that the differential $GZd\eta = d(BZ)$ is an increment of the segment BZ due to a vertical shift of point Z, as the buoyancy centre B moves horizontally, i.e. parallel to the waterplane. Hence, the classic formulation for the dynamic (righting) arm is obtained:

$$l_d = BZ - a \qquad (20)$$

where $a = B_0G$ is the height of the ship gravity centre G over buoyancy centre in an upright position (i.e. for $\eta = 0$). The said equation has a simple physical interpretation – the dynamic (righting) arm is equal to the vertical increment of the distance between the centre of gravity and centre of buoyancy. It is useful in checking accuracy of calculation of the *GZ*-curve.

The *GZ*-curve with free trim complies with the theorem of minimum potential energy, i.e. heeling of the ship (understood as rotation of the plane of rotation) by a given angle requires the least work. This is an important feature of the *GZ*-curve. The deflection of the ship from its longitudinal equilibrium is not possible without applying a trimming moment and doing additional work that increases its potential energy, which proves the above theory. Thus, the *GZ*-curve of a freely floating ship is at most equal to or smaller than that for a ship with fixed trim. Considering the above, the following holds for the dynamic arms of a freely floating ship and with fixed trim:

$$l_d = l_{dc} - \int_0^\Theta (\boldsymbol{e}\cdot\boldsymbol{r})d\Theta$$

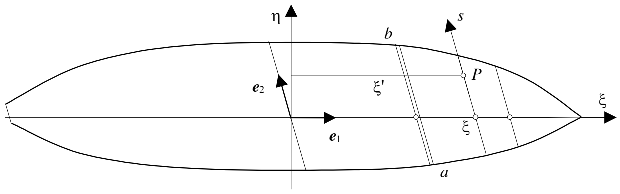

Figure 8. True view of the waterplane

where Θ is the trim angle for a given angle of rotation η of the plane of rotation, measured at a vertical plane. If one assumes that the longitudinal metacentric height H_L is constant in the course of trimming, then $\boldsymbol{e}\cdot\boldsymbol{r} = H_L\Theta$. Hence, $l_d \approx l_{dc} - ½H_L\Theta^2$. As can be seen, the sign of the trim has no meaning.

From the above equation two important conclusions can be drawn. Firstly, the greater the change of trim after balancing the ship, the lesser is the *GZ*-curve with free trim. Secondly, the *GZ*-curves of yet smaller arms would have to have yet larger trim changes, which is impossible due to the lack of the other equilibrium trim than that for a freely floating ship. By changing the trim, the centre of buoyancy permanently moves away from the plane of rotation. Hence, the *GZ*-curves with free trim are identical with *GZ*-curves of minimum stability. In other words, the longitudinal balance of the ship provides at the same time the *minimum* potential energy at a given heel angle.

It is worth remembering that a fixed trim θ, measured in the PS, does not mean that trim at a vertical plane Θ = *const*. Equation (3) implies that when θ = *const*, the angle Θ decreases to zero, when φ tends to 90°. This means that with an increase of the heel angle the difference between the *GZ*-curves at level keel and with fixed trim as in the initial position should vanish, which is supported also by numerical calculations, shown later.

As discussed earlier, the same *GZ*-curves for various reference axes and the same analytical angles φ and θ can be obtained, if the rotation axes $\boldsymbol{e}$ are the same, which is possible when the azimuth is accounted for. In addition, the dynamic (righting) arms l_d and the angles of rotation of the plane of rotation would be the same as well. The latter results from equation (19) for the dynamic arm l_d.

The notion of cross-curves of stability is fully valid for a freely floating ship, related to the reference axis Oz', normal to the initial waterplane. In other cases the *GZ*-curve should be corrected by a correction, accounting for the effect of trim change, induced by change of height of ship's centre of gravity (Pawlowski, 2013).

4.4 MOMENTS OF INERTIA

Given a hull of the ship, described in the *Oxyz* system, cut by an arbitrary plane. In ship statics the plane is the surface of the sea, whereas the cross-section itself is termed the waterplane. We want to find the principal moments of inertia for the said cross-section. They can be found *indirectly*, making use of moments of inertia for a projection of the cross-section (waterplane) on one of the co-ordinate planes (BP or PS), discussed in reference (Siemionov-Tian-Shansky, 1960), or *directly*, by calculating geometrical characteristics of the cross-section with the help of traces of the waterplane in the frame planes (Pawlowski, 1992b).

Moments of inertia will be found by the direct method. A typical cross-section of the hull, i.e. the waterplane, is shown in Figure 8. The ξ-axis coincides with the trace of water in the PS, whereas the η-axis is normal to the unit vectors $\boldsymbol{n}$ and $\boldsymbol{e}_1$. The origin of the η-axis is at the point of intersection of the z-axis with the trace of water in the PS. The traces of the waterplane in the frame planes, i.e. widths of the frames in the waterplane are oblique relative to the ξ-axis (trace of water in the PS); some of them are shown in Figure 8.

The angle between the unit vectors of the traces is equal to β. The ξ-axis divides a trace into two segments of lengths *a* and *b*; which can be directly measured in the frame planes. The quantities *a* and *b* have the meaning of the co-ordinates of the ends of the traces, measured along a trace. These co-ordinates are positive, if they are to the left of the ξ-axis, and negative, if they are to the right (Figure 8).

Introducing notation:

$$I_1 = \int (b-a)dx, \qquad J_{11} = \int (b-a)x dx,$$
$$I_2 = \int \tfrac{1}{2}(b^2 - a^2)dx, \qquad J_{12} = \int \tfrac{1}{2}(b^2-a^2)x dx,$$
$$I_3 = \int \tfrac{1}{3}(b^3 - a^3)dx, \qquad J_{21} = \int (b-a)x^2 dx,$$

where, in general $I_n \equiv J_{0n}$, finally we get the following expressions for the area of the waterplane, its static moments, cross-product and inertia moments:

$$\begin{aligned} A &= I_1 \sin\beta/\cos\theta, \\ M_\xi &= I_2 \sin^2\beta/\cos\theta, \\ M_\eta &= J_{11}\sin\beta/\cos^2\theta + \tfrac{1}{2}I_2 \sin 2\beta/\cos\theta, \\ D &= J_{12}\sin^2\beta/\cos^2\theta + I_3 \sin^2\beta\cos\beta/\cos\theta, \\ J_\xi &= I_3 \sin^3\beta/\cos\theta, \\ J_\eta &= J_{21}\sin\beta/\cos^3\theta + J_{12}\sin 2\beta/\cos^2\theta + \\ & \quad I_3\cos^2\beta\sin\beta/\cos\theta. \end{aligned} \tag{21}$$

Co-ordinates of the centre of gravity of the waterplane are as follows: $\xi_C = M_\eta/A$, $\eta_C = M_\xi/A$, whereas the central moments of inertia in the system ξ'η' shifted parallel to the waterplane centre of gravity (centre of floatation) are given by the parallel axes (Huygens–Steiner) theorem:

$$\begin{aligned} J_{\xi'} &= J_\xi - A\eta_C^2, \\ J_{\eta'} &= J_\eta - A\xi_C^2, \\ D' &= D - A\xi_C\eta_C. \end{aligned} \tag{22}$$

In further applications we need to know the central moments of inertia in the system ξ"η", where the ξ"-axis is parallel to the axis of rotation $\boldsymbol{e}$. For the reference axis *y*, the axis of rotation $\boldsymbol{e} = \boldsymbol{e}_1$ is parallel to the ξ-axis, therefore the system ξ"η" coincide with the system ξ'η'. For the reference axis *x*, the axis of rotation $\boldsymbol{e}$ is perpendicular to the trace of water on the frame planes $\boldsymbol{e}_2$. The ξ"-axis is therefore rotated with respect to the ξ-axis by an angle $\beta' = \beta - 90°$. For the reference axis *Oz*', normal to the initial waterplane, the axis of rotation $\boldsymbol{e}$ is inclined with respect to the ξ-axis at an angle β', given by the equation: $\cos\beta' = \boldsymbol{w}\cdot\boldsymbol{e}_1$, where $\boldsymbol{w}$ is a unit vector of the trace of water in the initial waterplane. It can be shown that the angle $\beta' > 0$, if $\theta > \theta_0$. The central moments in the system ξ"η", rotated by an angle β' relative to the system ξ'η', can be found from transformation of moments (22), given in the system ξ'η', discussed further down.

When the deck edge is immersed in water, the ξ-axis in Figure 8 (trace of PS in the waterplane), can go beyond the contour of the waterplane for large heel angles. The *s* co-ordinates of both ends of the trace of water at the frames have then the same sign. This has no particular meaning for calculations. It is worth knowing, however, that the ξ-axis can be defined by any buttock plane *y* = *const*, parallel to the PS, where the constant corresponds e.g. to the centre of projection of the trace of water in the midships section onto the BP. Selection of the ξ-axis is meaningless for the central moments of inertia, and hence, for the principal values of these moments.

4.5 METACENTRIC RADII. AXIS OF FLOATATION

In order to find an expression for metacentric radii, we have to resort to the theorem on shifted masses, and apply it to the wedges formed by rotation of the waterplane around the axis of floatation $\boldsymbol{f}$. It has the following form: $Vds = v|g_1g_2|$, where *ds* is the shift of the centre of buoyancy along the arc of the curve of centres of buoyancy, *V* is the volume displacement of the ship, g_1, g_2 are the centres of volume of the emerged and immersed wedge, *v* is the volume of one wedge, and $v|g_1g_2|$ is the static moment of the shifted wedge volume. This moment has two components: transverse, equal to $J_f d\alpha_1$, and longitudinal, equal to $D_f d\alpha_1$. Hence, $Vds = v|g_1g_2| = (J_f^2 + D_f^2)^{1/2} d\alpha_1$, where J_f and D_f are the central moments of inertia of the waterplane: transverse and cross-product, related to the axis of floatation $\boldsymbol{f}$. Introducing the notation: $J_s \equiv (J_f^2 + D_f^2)^{1/2}$, the above equation yields: $Vds = J_s d\alpha_1$. This can be written, as

$$ds = r_s d\alpha_1 \equiv r_B d\eta$$

where $r_s \equiv J_s/V$ is a proportionality factor between the shift of centre of buoyancy *ds* and the angle $d\alpha_1$, whereas the last identity results from definition of the metacentric radius. Considering that $d\alpha_1 \cos\chi = d\eta$, the following then results for the metacentric radius:

$$r_B = r_s/\cos\chi \tag{23}$$

As we can see, the metacentric radius r_B directly depends on the orientation of the floatation axis $\boldsymbol{f}$ relative to the axis of rotation $\boldsymbol{e}$. The knowledge of this axis accelerates the calculations. The metacentric radius r_B can be expressed in terms of the geometric characteristics of the waterplane in the system $\xi''\eta''$.

The central moments of the waterplane relative to the axis of floatation $\boldsymbol{f}$ are given by the expressions: $J_f = s + a_f$, where $s = ½(J_{\xi'} + J_{\eta'})$ is the centre of the inertia interval, and a_f is the radius of the inertia interval of the waterplane after rotation, given by the equations:

$$D_f = a''\sin 2\chi + D''\cos 2\chi, \qquad (24)$$
$$a_f = a''\cos 2\chi - D''\sin 2\chi,$$

$a'' = ½(J_\xi'' - J_\eta'')$ is the radius of the inertia interval before rotation (in the $\xi''\eta''$ system), whereas D'', J_ξ'', J_η'' are the moments of inertia of the waterplane in the central system $\xi''\eta''$, parallel to the axis of rotation $\boldsymbol{e}$ (Figure 9). Equations (24) are general; they say how the central moments change with a rotation of the system.

The centre of buoyancy moves in the rotation plane in parallel to the waterplane (water level). Therefore, the vector of displacement of centre of buoyancy is equal to $d\boldsymbol{r} = (\boldsymbol{n} \times \boldsymbol{e})\,ds$.

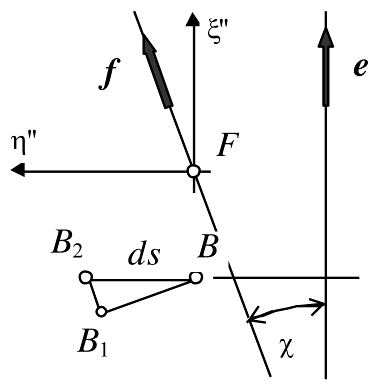

Figure 9. View from the top on the waterplane

Rotating the waterplane by an angle $d\alpha_1$, the transverse component of the buoyancy centre displacement BB_1, relative to the axis of floatation (Figure 9) is proportional to J_f, whereas the longitudinal component B_1B_2 is proportional to D_f. We want the resultant displacement to be normal to the direction of the heeling moment (axis of rotation $\boldsymbol{e}$). To be so, the angle B in Figure 9 has to be equal to χ, which results from the property of angles, whose arms are normal respectively. Hence, the angle of inclination of the axis of floatation relative to the axis of rotation has to satisfy the equation:

$$\tan\chi = D_f / J_f \qquad (25)$$

The angle χ has the same sign as that of the waterplane product of inertia (in Figure 9 it is positive). It should be remembered that moments D_f and J_f are also dependent on the angle χ, which converts the above formulation to an equation. Substituting $J_f = s + a_f$, equation (25) will take the form:

$$D_f - (s + a_f)\tan\chi = 0$$

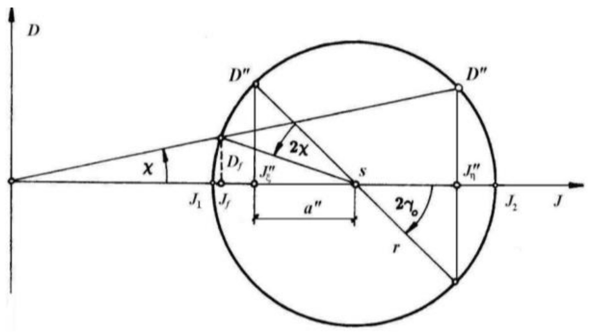

Figure 10. Mohr's circle for the waterplane

The quantities D_f and a_f, given by equation (24), represent a parametric equation of the Mohr's circle (Figure 10). Substituting them to the above equation yields:

$$r\sin(2\gamma + 2\chi) - [s + r\cos(2\gamma + 2\chi)]\tan\chi = 0$$

where $r = (a^2 + D^2)^{1/2}$ is the radius of the Mohr's circle, independent of the orientation of the central system, the phase $2\gamma_0 = \tan^{-1}(D''/a'')$, the angle $2\gamma = 2\gamma_0$, if $a'' > 0$, otherwise $2\gamma = 2\gamma_0 + 180°$; a'' and γ_0 are negative in Figure 10. The secant sD'' defines the location of the ξ''-axis in the Mohr's circle plane, while the principal axis ξ_1 coincides with the abscissa axis but it has the opposite direction. The angle between them equals: $2\gamma' = 180° - 2\gamma = -2\gamma_0$. The principal moments are given by the equation: $J_{2,1} = s \pm r$.

When $\cos 2\chi$ and $\sin 2\chi$ in the above equation are expressed by $\tan\chi$, it can be reduced to a simple equation: $D'' = (s - a'')\tan\chi$. Hence,

$$\tan\chi = D''/J_\eta'' \qquad (26)$$

where $|\chi| \le \sin^{-1}(r/s)$, and $|\chi| \le |\gamma_0| < 45°$, which is seen in Figure 11. The axis of floatation $\boldsymbol{f}$ is between the axis of rotation $\boldsymbol{e}$ and the principal axis if inertia of the waterplane ξ_1. Triangle J_1sD'' is an isosceles triangle, in which the exterior angle equals $2\gamma'$. Thence, secant J_1D'' is inclined relative to the abscissa axis at the angle $\gamma' = -\gamma_0$.

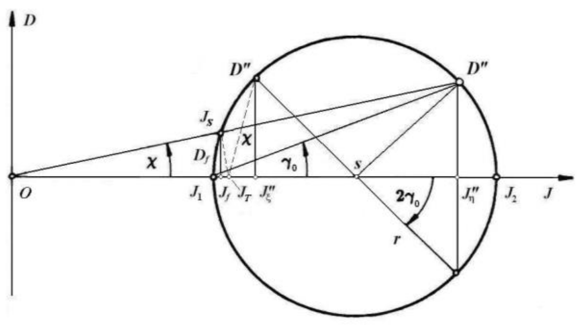

Figure 11. Principal direction and transverse moment of waterplane inertia J_T

Equation (26) has a simple physical interpretation. The directed angle $\boldsymbol{f}d\alpha_1$ has two components in the system $\xi''\eta''$: the axial $d\eta$ and normal $d\tau$. Rotation of the waterplane around the axis $\boldsymbol{e}$ yields a longitudinal displacement of the centre of buoyancy, proportional to $D''d\eta$, which must be compensated by trimming $J_\eta''d\tau$. Hence, $D''d\eta = J_\eta''d\tau$. Therefore, $d\tau/d\eta = D''/J_\eta''$, where the ratio of differentials $d\tau/d\eta = \tan\chi$.

Strictly speaking, the static moment of the shift of volume displacement in the longitudinal direction $D''d\eta$ has to be compensated by the trimming moment $VH_L d\tau$, where VH_L is the longitudinal coefficient of stiffness. Hence: $D''d\eta = VH_L d\tau$. Thus: $d\tau/d\eta = D''/VH_L$, which yields an improved equation (26), provided in publication [5]:

$$\tan\chi = D''/VH_L \qquad (27)$$

where V is the volumetric displacement of the ship, $H_L = R_L - BZ$ is the longitudinal metacentric height, $R_L = J_\eta''/V$ is the longitudinal metacentric radius, while $BZ = -\boldsymbol{r}\cdot\boldsymbol{n}$ is the height of the gravity centre above the centre of buoyancy (Figure 2–4). Hence, the coefficient of stiffness $VH_L = J_\eta'' - V\cdot BZ$. In the case of conventional ships, the term $BZ\cdot V$ is negligibly small in comparison to the longitudinal moment of inertia of the waterplane J_η'', therefore equation (26) is practically the same as equation (27). In the case of platforms and for large heel angles, this term cannot be neglected.

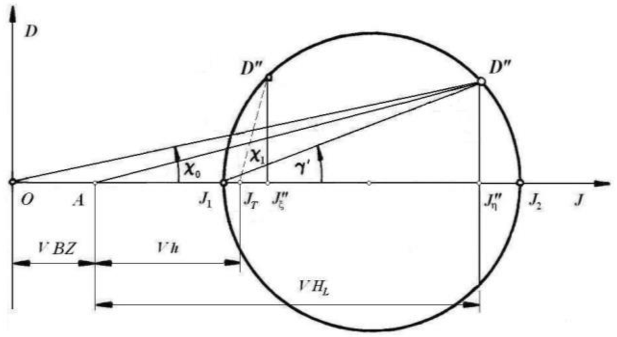

Figure 12. Mohr's circle and stability characteristics

Geometrical interpretation of solution (27), denoted by χ_1, is shown in Figure 12. The solution of equation (26) is denoted by χ_0. Straight line AD'' is inclined at the angle χ_1. It is clear that $\chi_1 > \chi_0$, which decreases the moment of inertia of the waterplane J_T, thereby decreases the metacentric radius r_B. It can be seen also in Figure 12 that $\chi_1 < \gamma'$.

The shift of the centre of buoyancy can be analysed also in the principal system $\xi_1\eta_1$, rotated by the angle γ' relative to the central system $\xi''\eta''$ (Figure 13). The directed angle $\boldsymbol{f}d\alpha_1$ has two components in the system $\xi_1\eta_1$: the axial one $\Delta\alpha_1\cos(\gamma' - \chi)$, and normal $-\Delta\alpha_1\sin(\gamma' - \chi)$, causing trim by bow. The transverse component of the buoyancy centre displacement BB_1 is proportional to $J_1\Delta\alpha_1\cos(\gamma' - \chi)$, whereas the longitudinal one B_1B_2 is proportional to $J_2\Delta\alpha_1\sin(\gamma' - \chi)$. The resultant displacement has to be normal to the axis of rotation $\boldsymbol{e}$. To be so, the angle B in Figure 13 has to be equal to γ', whose $\tan\gamma' = B_1B_2/BB_1$, leading to a simple equation for the angle χ:

$$(J_2/J_1)\tan(\gamma' - \chi) = \tan\gamma' \qquad (28)$$

where J_1 and J_2 are the principal moments of inertia of the waterplane. It follows from equation (28) that the angle χ is inside the interval $\langle 0, \gamma'\rangle$. In other words, the axis of floatation $\boldsymbol{f}$ is between the principal axis of inertia ξ_1 and the axis of rotation $\boldsymbol{e}$. The three axes coincide only, when the angle $\gamma' = 0$. Equation (28) can be improved replacing the principal moments by the coefficient of stiffness.

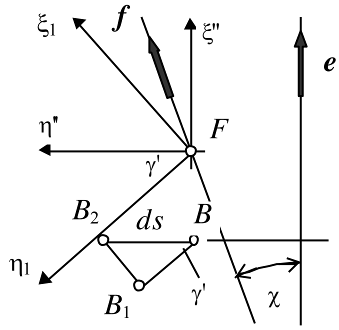

Figure 13. Components of shift of centre of buoyancy in the principal co-ordinate system $\xi_1\eta_1$

The knowledge of the angle χ defines the direction of the axis of floatation $\boldsymbol{f}$. The unit vector of this axis is as follows: $\boldsymbol{f} = \boldsymbol{e}\cos\chi + (\boldsymbol{n}\times\boldsymbol{e})\sin\chi$.

Examine now the transverse moment of inertia of the waterplane J_T that defines the metacentric radius $r_B \equiv J_T/V$. Multiplying equation (23) by the volumetric displacement V, and accounting for equation (26), the following is obtained:

$$\begin{aligned} J_T = J_s/\cos\chi &= (J_f^{\,2} + D_f^{\,2})^{1/2}/\cos\chi \\ &= J_f[1+(D_f/J_f)^2]^{1/2}/\cos\chi \\ &= J_f(1+\tan^2\chi)^{1/2}/\cos\chi = J_f/\cos^2\chi \end{aligned}$$

Substituting $J_f = s + a_f$, where $s = ½(J_\xi'' + J_\eta'')$ is the centre of the inertia interval of the waterplane, while a_f is the radius of the inertia interval of the waterplane after rotating by an angle χ, given by equation (24), the following is obtained:

$$\begin{aligned} J_T = (s + a_f)/\cos^2\chi &= (s + a''\cos 2\chi - D''\sin 2\chi)/\cos^2\chi \\ &= s/\cos^2\chi + a''(2 - 1/\cos^2\chi) - 2D''\tan\chi \\ &= (s - a'')(1 + \tan^2\chi) + 2a'' - 2D''\tan\chi \\ &= s + a'' + (s - a'')\tan^2\chi - 2D''\tan\chi \\ &= J_\xi'' + J_\eta''\tan^2\chi - 2D''\tan\chi \end{aligned}$$

Taking into account equation (26), we get the equation:

$$J_T = J_\xi'' - D''\tan\chi \qquad (29)$$

from which it follows that $J_T \leq J_\xi''$. It means that balancing the ship decreases the transverse moment of inertia of the waterplane J_T, and also the metacentric radius r_B, which in turn causes a reduction of the righting arm – a conclusion consistent with the foregoing considerations that balancing the ship decreases the stability. The expression $J_T = J_s/\cos\chi = J_\xi'' - D''\tan\chi$ has a simple interpretation, shown in Figure 11 and Figure 12.

Equation (29) can be derived directly. A rotation of the waterplane around the axis $\boldsymbol{e}$ yields a transverse shift of the centre of buoyancy, proportional to $J_\xi''d\eta$. On the other hand, balancing the ship decreases this shift by $D''d\tau$. The resultant shift, by definition, is proportional to $J_T d\eta$. Hence: $J_T d\eta = J_\xi''d\eta - D''d\tau$. Dividing it by $d\eta$ yields equation (29).

Equations (25) to (29) were derived assuming that $\boldsymbol{e}\cdot d\boldsymbol{r} = 0$, i.e. that the displacement of the centre of buoyancy $d\boldsymbol{r}$ is strictly perpendicular to the axis of rotation $\boldsymbol{e}$. However, for a freely floating ship this is not the case. Note that when the ship is heeled the trim has to be changed to balance the ship, which changes orientation of the rotation axis $\boldsymbol{e}$ relative to the ship.

Differentiating $\boldsymbol{e}\cdot\boldsymbol{r} = 0$ we get: $\boldsymbol{e}\cdot d\boldsymbol{r} = -d\boldsymbol{e}\cdot\boldsymbol{r}$, i.e. in the co-ordinate system fixed to the ship the displacement of centre of buoyancy is not strictly normal to the axis of rotation. This should be intuitively obvious: since the centre of buoyancy has to remain all the time in the plane of rotation, which changes its orientation relative to the inclining ship, the displacement of centre of buoyancy has to be oblique to it.

When the axis of floatation $\boldsymbol{f}$ is known, it is easy to find new analytical angles φ and θ, describing orientation of the ship relative to the water at the new angle of heel. Namely, rotating the waterplane by an angle $\Delta\alpha_1$, the unit vector $\boldsymbol{n}$ rotates around the axis of floatation by the angle $\Delta\alpha_1$. Hence, the new unit vector $\boldsymbol{n}_1$ is as follows:

$$\boldsymbol{n}_1 = \boldsymbol{n}\cos\Delta\alpha_1 + (\boldsymbol{f}\times\boldsymbol{n})\sin\Delta\alpha_1$$

Knowing new unit vector $\boldsymbol{n}$ ($\equiv \boldsymbol{n}_1$), the new analytical angles, corresponding to the new unit vector can be easily obtained from the first formulation in equation (7). Namely, $\tan\theta = -n_x/n_z$, whereas $\tan\varphi = -n_y/n_z$. The knowledge of new angles of waterplane inclination largely speeds up the process of finding the correct location of the centre of buoyancy at a new angle of heel φ, ϕ or α, depending on the line of nodes. The equation of new waterplane at first iteration is as follows:

$$n_x(x - x_F) - n_y(y - y_F) - n_z(z - z_F) = 0$$

where x_F, y_F, z_F are co-ordinates of the previous centre of floatation F, whereas $(n_x, n_y, n_z) = \boldsymbol{n}_1$ are components of the new unit vector $\boldsymbol{n}$. Knowing the equation of the waterplane it is necessary to check by iterations, if the ship displacement $V = const$ is conserved, and if the ship is longitudinally balanced, i.e. if the equation $\boldsymbol{e}\cdot\boldsymbol{r} = 0$ is satisfied. If not, then the waterplane should be shifted in the normal direction by a distance $\Delta n = -\Delta V/A_{WL}$, and the trim angle Θ, θ or ϑ, depending on the line of nodes, should be corrected. If the centre of buoyancy is in front of the plane of rotation ($l_e > 0$), the trim angle should be somewhat decreased, by rotating the waterplane around the axis η'' (Figure 9) in positive direction by an angle $\Delta\tau = l_e/H_L$, where H_L is the longitudinal metacentric height. Depending on the reference axis (line of nodes) the change of trim angle is as follows:

$$-\Delta\tau = \Delta\theta\cos\phi = -\Delta\vartheta\sin\alpha = \Delta\Theta \tag{30}$$

which results from the vector properties of small rotations, i.e., a projection of the directed angle of trim on the horizontal plane (Figure 14). Substituting $\Delta\tau = l_e/H_L$, the following is obtained:

$$-l_e = H_L\Delta\Theta = (H_L\cos\phi)\Delta\theta = -(H_L\sin\alpha)\Delta\vartheta$$

The multipliers of the trim changes are the coefficients of stiffness with respect to trim, i.e., the longitudinal metacentric heights. Except the vertical trim, in the case of oblique trims the metacentric heights are incomplete, as they neglect the effect of the horizontal change of trim.

Calculations of the GZ-curve can be significantly accelerated, if they are based on the Kryłov–Dargnies' method, modified for a freely floating ship, utilising the properties of equi-volume waterplanes for such a ship, unknown in literature. In a finite interval of the angle of rotation $\Delta\eta$ equi-volume waterplanes roll over the surface of a certain non-circular cone, the parameters of which can be predicted in advance [5]. The rolling waterplanes are tangent to the cone along the instantaneous axis of floatation $\boldsymbol{f}$.

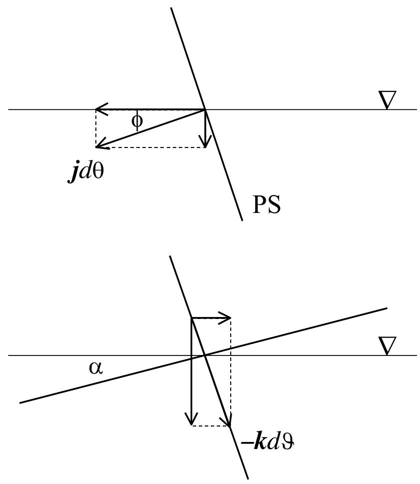

Figure 14. Positive change of oblique trim

The axis of the cone is inclined relative to the waterplanes by an angle ε determined by the following expression: $\sin\varepsilon = d\chi/d\alpha_1$. Its derivation is elementary. When a cone rolls over a plane with no slip, the base of the cone moves along an arc of length $l\chi = r\alpha$. Hence, $\chi/\alpha = r/l = \sin\varepsilon$, where χ is the angle of rotation of the cone in the plane, α

is the angle of rotation of the cone around its own axis of symmetry, r is the radius of the base, and l is the length of the generatrix. In kinematics, the said cone, over which equi-volume waterplanes roll over, is an example of a ruled *fixed axode*, whereas rolling waterplanes – a *moving axode*.

When the angle $d\chi > 0$ is positive the cone is located above the waterplanes, if not – below. The apex of the cone is located at a distance from the generatrix l from the centre of floatation F, given by the equation $l = -d\eta'_F/d\chi$, where $d\eta'_F$ is the displacement of centre of floatation normal to the axis floatation (when $l > 0$, the apex is located in the direction of the bow). Taking into account that $d\eta'_F = r_F d\alpha_1$ one obtains:

$$l = -r_F\, d\alpha_1/d\chi = -r_F/\sin\varepsilon$$

where $r_F = dJ_f/dV$ is a differential metacentric radius (radius of curvature of the curve of centres of floatation). This formulation shows that the radius of the cone base at the level of the centre of floatation is equal to the differential metacentric radius.

4.6 MECHANISM OF EQUI-VOLUME INCLINATIONS

An infinitesimal rotation of the waterplane around the axis of floatation $\boldsymbol{f}$ can be regarded as resulting from two rotations: ship's rotation by an angle $d\eta$ around the axis ξ'', parallel to the axis of rotation $\boldsymbol{e}$, and ship's rotation by angle $d\tau$ around the axis η'', normal to the axis of rotation $\boldsymbol{e}$ (Figure 9). Hence, the directed angle $\boldsymbol{f}d\alpha_1$ has two components in the system $\xi''\eta''$, equal to the two said elementary rotations: $\boldsymbol{f}d\alpha_1 = (d\eta,\ d\tau)$.

The directed angle $\boldsymbol{f}d\alpha_1$ is inclined at an angle χ to the rotation axis $\boldsymbol{e}$ (Figure 9). Positive angle χ corresponds to *positive* normal component of $d\tau$, whereas the change of trim is *negative* (by stern), therefore the normal component has to be taken with an opposite sign. Projection of $d\alpha_1$ on the rotation axis yields the axial component $d\eta = d\alpha_1\cos\chi$, which implies that the elementary rotation $d\eta > 0$ (the angle $\chi < 45°$). Resorting to the relationships inherent for rectangular triangles, the normal component $d\tau$ can be written in two ways:

$$d\tau = d\alpha_1 \sin\chi = d\eta \tan\chi \qquad (31)$$

The above equation indicates that: 1) the more deflected the axis of floatation from the rotation axis, the greater changes of ship trim during inclinations, which is intuitive; 2) when $\chi = 0$, i.e. when $\boldsymbol{e} = \boldsymbol{f}$, the ship trim does not change, as for a ship with fixed trim; 3) as $d\tau$ is finite the trim angle $\Theta < 90°$ cannot reach 90°. In other words, in the course of heeling the rig cannot "rear", and 4) from equation (30) it follows that for $\phi = 90°$ (the PS is then horizontal) $d\tau = 0$. We will see later that achieving the angle $\phi = 90°$ is impossible by a free-floating ship.

In the case of the reference axes y and z, the rotation of the reference planes around normal vectors, associated with trimming, equals to $\boldsymbol{j}d\theta$ or $-\boldsymbol{k}d\vartheta$ has also a vertical component $d\psi$, which equals the rotation (the change of orientation) of the ship in the sea surface. In the case of the PS, it equals $d\theta\sin\phi$, and in the case of the BP, it equals $-d\vartheta\cos\alpha$ (Figure 14). Hence,

$$d\psi = -d\theta \sin\phi = -d\vartheta \cos\alpha. \qquad (32)$$

In both cases, the vertical component of rotation of the plane of rotation is directed downwards, which means that rotation of the ship in the horizontal is clockwise. If this rotation were neglected, the trim would change the azimuth.

Considering equations (30) and (31) the differential $d\psi$ can be expressed in terms of an increase of the heel angle $d\eta$. Namely, $d\psi = d\tau\tan\phi = d\tau\cot\alpha$, where $d\tau = d\eta\tan\chi$ is a rotation of the ship in the horizontal. The angles of rotations of the PS or the initial waterplane around the trace of water have no vertical components, as they are directed horizontally (Figure 1).

A different situation occurs in the case of the reference axis x: a change of the trim angle, as a vector, is directed horizontally, therefore it has no vertical component (Figure 2). However, the angle of rotation of the midships around its normal $-\boldsymbol{i}d\varphi$ has a horizontal component $-d\varphi\cos\Theta$, and vertical: $-d\varphi\sin\Theta$. Rotations of the waterplane relative to the ship have the opposite sign: $(d\varphi\cos\Theta,\ d\varphi\sin\Theta)$. The horizontal component is the angle of rotation of the waterplane relative to the ship. Hence, $d\eta = d\varphi\cos\Theta$. The vertical component $d\psi = d\varphi\sin\Theta = d\eta\tan\Theta$ is a change of orientation of the rotation axis $\boldsymbol{e}$ relative to ship. When in an upright position the ship is trimmed, the angles φ and Θ are replaced by φ' and Θ', and the angles α and ϑ by α' and ϑ'.

The rotation of the ship in the horizontal by an angle $d\psi$, induced by trimming (balancing) the ship, has no *direct* effect on calculating the GZ-curve. In particular, it has no effect on the orientation of the axis of floatation $\boldsymbol{f}$ in the ship system. Hence, if for a new waterplane the angle χ changes by $d\chi$ the new floatation axis will rotate relative the previous one by an angle $d\chi$, as rotation of the ship in the horizontal plane does not change the waterplane. When the angle $d\chi > 0$ is positive, the new floatation axis $\boldsymbol{f}$ shifts towards the heel, i.e. it departs from the rotation axis $\boldsymbol{e}$.

If ship heel is increased by $d\eta$, the displacement of centre of buoyancy, normal to the plane of rotation, is proportional to $D''d\eta$, where D'' is the product of inertia of the waterplane in the $\xi''\eta''$ system (Figure 9). The said displacement must be compensated by trim $J_\eta''d\tau$. Equating them to each other one gets $d\tau = (D''/J_\eta'')\,d\eta$. Hence, $d\tau/d\eta = D''/J_\eta''$. Taking into account equation (31), the above yields equation (26). A more exact solution can be obtained by using the metacentric formulation for $d\tau = (D''/VH_0)d\eta$, where $H_0 \equiv GM_L = BM_L - BZ$ is the longitudinal metacentric height. As $\tan\chi = d\tau/d\eta$, the above yields equation (27).

The righting arm is given by equation (17). In order to make use of it, for given heel angle $\eta = const$ and given volume displacement $V = const$ we have to know the trim at which the ship is balanced, i.e. $\boldsymbol{e} \cdot \boldsymbol{r} = 0$. Usually, we find it by an iterative method. This process can be accelerated, if the change of the longitudinal component of the righting arm $dl_e = d(\boldsymbol{e} \cdot \boldsymbol{r}) = d\boldsymbol{e} \cdot \boldsymbol{r} + \boldsymbol{e} \cdot d\boldsymbol{r}$, induced by trim is known.

The change of the axis of rotation $d\boldsymbol{e}$ in the ship hull system due to trimming can be easily worked out with the help of Figure 2–4. In the first case the change results from vertical rotation of the unit vector $\boldsymbol{e}$ by an angle $d\Theta$, in the second – by an angle $d\theta$ in the PS, and in the third case – by an angle $d\vartheta$ in the initial waterplane. Hence,

$$\begin{aligned} d\boldsymbol{e} &= \boldsymbol{n}d\Theta, \\ d\boldsymbol{e} &= (\boldsymbol{e} \times \boldsymbol{j})d\theta, \\ d\boldsymbol{e} &= -(\boldsymbol{e} \times \boldsymbol{k})d\vartheta. \end{aligned} \tag{33}$$

The above equations can be easily checked directly by differentiating $\boldsymbol{e}$ with respect to appropriate trim angle Θ, θ or ϑ, depending on the reference axis. Thus,

$$\begin{aligned} d\boldsymbol{e} \cdot \boldsymbol{r} &= \boldsymbol{r} \cdot \boldsymbol{n}d\Theta = -BZd\Theta, \\ d\boldsymbol{e} \cdot \boldsymbol{r} &= \boldsymbol{r} \cdot (\boldsymbol{e} \times \boldsymbol{j})d\theta = \boldsymbol{r} \cdot \boldsymbol{e}_Z = r_Z d\theta \\ &= -(BZ\cos\phi - l\sin\phi)d\theta, \\ d\boldsymbol{e} \cdot \boldsymbol{r} &= -\boldsymbol{r} \cdot (\boldsymbol{e} \times \boldsymbol{k})d\vartheta = \boldsymbol{r} \cdot (\boldsymbol{k} \times \boldsymbol{e})d\vartheta \\ &= (BZ\sin\alpha + l\cos\alpha)d\vartheta, \end{aligned}$$

where BZ is a vertical distance between the ship centre of gravity and centre of buoyancy (Figure 2), $\boldsymbol{e}_Z \equiv \boldsymbol{e} \times \boldsymbol{j}$ is the unit vector of the OZ axis, fixed to the plane of rotation; the said axis is the edge of intersection between the PS and rotation plane (Figure 3), r_Z is a projection on the axis OZ of the radius vector $\boldsymbol{r}$ of the centre of buoyancy relative to the ship centre of gravity, and $\boldsymbol{r} \cdot (\boldsymbol{k} \times \boldsymbol{e})$ is a projection of $\boldsymbol{r}$ on the edge of intersection between the plane of rotation and the initial waterplane. The second relation results from a projection of the segment BZ on the OZ-axis, deviated from the vertical by the angle ϕ (Figure 3), and the third one – from a projection of BZ on the axis Oz', deviated from the vertical by the angle α' (Figure 4).

In the case of the reference axis Ox', the second contribution to the change dl_e is given by the relation: $\boldsymbol{e} \cdot d\boldsymbol{r} = R_L d\Theta$, where R_L is the longitudinal metacentric radius, which follows from the preceding considerations. For other reference axes, the vertical change of the trim angle is given by equation (30).

Additionally, we have to account for the effect of rotation of the ship in the horizontal on the displacement of the centre of buoyancy relative to the (stationary) plane of rotation. It equals to $-ld\psi$, which directly results from Figure 3–4, where $d\psi$ is the trim induced rotation of the ship in the horizontal, given by equation (32), and $l \equiv GZ$ is the righting arm. When $d\psi < 0$, the rotation is clockwise, while the displacement of the centre of buoyancy is positive, i.e. in the bow direction. For the reference axis x', $d\psi = 0$, since the vertical change of trim does not cause any rotation in the sea surface (Figure 2); the said rotation occurs only during oblique trimming (Figure 3–4).

Hence, combining the said contributions, depending on the reference axis the following is obtained for change of the trimming arm dl_e:

$$\begin{aligned} dl_e &= (R_L - BZ)d\Theta, \\ dl_e &= [R_L\cos\phi - (BZ\cos\phi - l\sin\phi) + l\sin\phi]d\theta, \\ dl_e &= [-R_L\sin\alpha + (HF\sin\alpha + l\cos\alpha) + l\cos\alpha]d\vartheta. \end{aligned}$$

After simplifications, we get eventually:

$$\begin{aligned} dl_e &= H_L d\Theta, \\ dl_e &= (H_L\cos\phi + 2l\sin\phi)d\theta, \\ dl_e &= (H_L\sin\alpha - 2l\cos\alpha)(\mp d\vartheta). \end{aligned} \tag{34}$$

In the third case, we have to pay attention to the sign of α. When the heel is to portside ($\alpha > 0$), a positive increase of the twist angle $d\vartheta$ means trimming by aft, i.e. the change $dl_e < 0$ is negative. Hence, $d\vartheta$ has to be taken with the opposite sign. When the heel is to starboard ($\alpha < 0$), a positive increase of the twist angle $d\vartheta$ produces the change dl_e consistent with the sign of $d\vartheta$. In other words, the expression for $H_{L\vartheta}$ changes the sign when $\alpha < 0$.

These equations allow for quick finding of the equilibrium trim. The expressions in the parentheses represent a derivative of the longitudinal component of the righting arm l_e relative to the respective trim angle, that is, the *longitudinal metacentric height* for a given reference axis $H_{L\Theta}$ $H_{L\theta}$ i $H_{L\vartheta}$, understood as the stiffness relative to a respective trim angle. The first one is the classic longitudinal metacentric height $H_{L\Theta} \equiv H_L$ for vertical trims. In the case of oblique trims, the longitudinal metacentric height depends additionally on the righting arm $l \equiv GZ$.

In the course of heeling the longitudinal metacentric height varies. When it becomes negative, it means the lack of longitudinal balance, ipso facto, the lack of opportunity for determining the righting arm. This phenomenon is termed as *fading stability*. This phenomenon does not occur when doing calculations with fixed trim – the GZ-curve is defined at each heel angle.

In an upright position $H_{L\Theta} = H_{L\theta} = H_L$, and $H_{L\vartheta} = -2l_0$. When $l_0 = 0$, where l_0 is the righting arm in an upright position, the longitudinal metacentric height is an even function of the heel angle. When $l_0 \neq 0$, i.e., when an initial heel occur, in the case of the reference axis Oz' the GZ-curve is indefinite in some one-sided neighborhood of zero. For two other reference axes, the GZ-curve is continuous around zero. When $\alpha \to 90°$, $H_{L\vartheta} \to H_L$ tends to the longitudinal metacentric height, as for the reference axis Ox', whereas $H_{L\theta}$ tends to negative values. It means that in some vicinity of the angle $\phi = 90°$ the GZ-curve related to the reference axis Oy is indefinite.

The expression for $H_{L\vartheta}$ allows for the estimation of the external end of the interval, in which the *GZ*-curve for the axis Oz' is indefinite. Equation (34) yields:

$$\tan\alpha = 2l/H_L \tag{35}$$

The above angle can be expressed in terms of the initial heel α_0. Assuming that $\alpha_0 = -l_0/h_0$, where h_0 is the initial metacentric height, we get: $\alpha = -2\alpha_0 h_0/H_L$. As we can see, the length of the interval with faded stability is proportional to the angle of initial heel, located on the other side of zero, starting exactly at zero. For conventional ships the said interval is imperceptible. However, it is characteristic for semisubmersible platforms, particularly for jack-up rigs, where the longitudinal metacentric height is relatively small and the righting arms relatively large. For inclinations in the direction of the initial heel, the *GZ*-curve is definite at each point.

The angle χ, given by equation (27), describing orientation of the axis of floatation $\boldsymbol{f}$ relative to the axis of rotation $\boldsymbol{e}$, was obtained without accounting for the rotation of the ship in the horizontal plane. The said angle affects the transverse metacentric radius $r_B = J_T/V$ through the transverse moment of inertia of the waterplane J_T, given by equation (29). The improvement of the relation for the angle χ is simple. The rotation of the ship by an angle $d\eta$ yields not only the static moment of shifting the displacement in the longitudinal direction, equal to $D''d\eta$, but yields also the rotation in the horizontal by an angle $d\psi = d\varphi\sin\Theta$, directed upwards, if the ship is trimmed by bow. The said rotation moves the centre of buoyancy away from the plane of rotation towards the aft by $ld\psi$. The resultant change of the static moment has to be compensated by a trimming moment $VH_L d\tau$. Hence:

$$D''d\eta - Vld\psi = VH_L d\tau,$$
$$D'' - Vld\psi/d\eta = VH_L d\tau/d\eta.$$

Taking into account that $d\psi/d\eta = \tan\Theta$, and $d\tau/d\eta = \tan\chi$, the following is obtained:

$$\tan\chi = (D'' - Vl\tan\Theta)/VH_L \tag{36}$$

The above equation is valid for the reference axis Ox. When the ship has an initial trim, the angle Θ is replaced by Θ'. If $\tan\Theta$ is negligible, the above reduces to equation (27).

In the case of the two remaining reference axes, the elementary rotation of the ship $d\eta$, equal to $d\phi$ or $d\alpha$, there is no a vertical component. Therefore, the static moment of shifting the displacement in the longitudinal direction $D''d\eta$ has to compensated by trimming $-Vdl_e$, where dl_e is given by equation (34). Hence: $D''d\eta$ has to be equal to $-VH_{L\theta}d\theta$ or $VH_{L\vartheta}d\vartheta$. Accounting for equations (30), the following is obtained:

$$\tan\chi = \cos\phi D''/VH_{L\theta} \tag{37}$$
$$\tan\chi = -\sin\alpha D''/VH_{L\vartheta}$$

When the ship has an initial trim, the angle α is replaced by α'.

5. GZ-CURVE OF MINIMUM STABILITY

As previously mentioned, most heeling moments acting on the ship, including the wind heeling moment, are parallel to the PS, therefore a free-floating ship assumes the position in which the trace of water in the PS is normal to the rotation plane. In the case of platforms arbitrarily orientated to the wind, the wind generated heeling moment is parallel to the wind impact plane, perpendicular to the wind direction in an upright position, fixed to the platform. Hence, the heeling moment is parallel to the trace of water in the impact plane, whereas the rotation plane is perpendicular to the said trace. A question then arises which position does the ship assume when the direction of the moment is not related to the ship?

In order to answer unequivocally this question, the mechanism of inclining the ship in the case of a free heeling moment must be known. In such a case the ship assumes a position in which the potential energy is minimal, i.e., the work required to incline the ship is lowest. We know that this property has a freely floating ship, longitudinally balanced. For a given heel angle there is only one equilibrium position $\boldsymbol{e}\cdot\boldsymbol{r} = 0$, corresponding to minimum energy, independent of the reference axis.

The work is proportional to the dynamic (righting) arm, hence the minimum of potential energy corresponds to the minimum of the dynamic arm ld, given by equation (19), valid in any case. From the classic ship theory it is known that the dynamic arm depends on the run of metacentric radii in function of the heel angle, which for a freely floating ship means in function of the rotation angle η of the rotation plane. Hence, in a general case:

$$l_d = \int_0^\eta r_B\sin(\eta - \nu)d\nu - a(1 - \cos\eta), \tag{38}$$

where ν is a dumb variable of integration, varying from 0 to η (given angle of rotation of the rotation plane), $r_B = J_T/V$ is the metacentric radius in the rotation plane, whereas $a = BG$ is a constant, equal to the distance between the centre of buoyancy and centre of gravity at an upright position. It is obvious that the minimum is dependent on the integrand in equation (38), which is minimal for the least metacentric radii in function of the rotation angle. And this happens, when the angle χ by which the axis of flotation $\boldsymbol{f}$ is deviated from the axis of rotation $\boldsymbol{e}$, given by equation (36) or (37), is minimal. This happens, when the azimuth $\psi = 0$, and when the ship is longitudinally balanced.

In other words, the ship inclines around the instantaneous axis of flotation $\boldsymbol{f}$. That is, it rolls over a non-circular cone (a fixed axode), tangent to the waterplane along a generatrix, coinciding with the axis of flotation. Centre of buoyancy B moves in the ship system along a spatial curve, lying on the surface of a horizontal cylinder of varying radius of curvature, forming a kind of helix, intersecting at a certain angle the stationary rotation plane (the large circle in Figure 2–4). At each point the said line has a tangent, parallel to the respective waterplane (Figure 15). The righting arm $l \equiv GZ$ is a chord of the arc, created by the projection of the curve of centres of buoyancy on the sea surface, the axis of rotation ***e*** is perpendicular to the righting lever l, inclined at an angle χ with respect to the axis of floatation ***f***, while the dynamic arm l_d is an increase of the vertical distance between points G and B.

The righting arm GZ lies at the vertical rotation plane, stationary in space, passing through points G and B. The centre of buoyancy moves in the rotation plane along a flat curve of centres of buoyancy, whose metacentric radius $r_B = J_T/V$, where J_T is the transverse moment of inertia of the waterplane, given by equation (29), dependent on the waterplane geometrical characteristics in the system related to the axis of rotation ***e***.

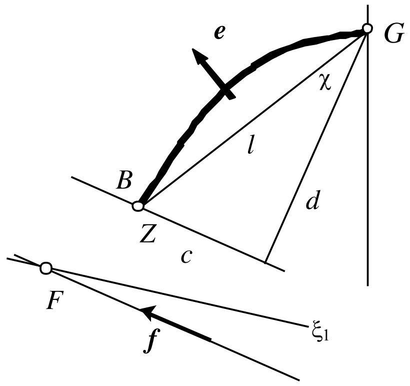

Figure 15. Projection of curve of centres of buoyancy on the waterplane

The least GZ-curve, termed the *GZ-curve of minimum stability* is identical with the curve for a freely floating ship, related to the axis Oz'. The righting arm for a given heel angle corresponds to the first zero of the curve $l_e \equiv \boldsymbol{e} \cdot \boldsymbol{r}$ in function of the azimuth ψ (Figure 22). In this point the absolute minimum of potential energy occurs (minimum of the dynamic arm l_d), clearly seen in the said figure, consistent with the meaning of this curve. The axis of floatation ***f*** is located between the axis of rotation ***e*** and the principal axis of inertia of the waterplane ξ_1, as discussed in section 4.5.

GZ-curves for the reference axes Ox' and Oy have the least values for the azimuth $\psi = 0$. They have the same area between the angle of equilibrium and angle of vanishing stability, as in the case of the axis Oz'. Therefore, they can also be regarded as the curves of minimum stability. The direction of the righting moment for the said reference axes, described by the axis of rotation ***e***, is stationary in space. The same applies to the reference axis Oz', though it is said literature that the righting moment of the curve of minimum stability has a varying direction in space, which is not true. The plane of rotation (the large circle in Figure 2–4) is stationary in space, and the same applies to the axis of rotation ***e***, normal to it.

For the reference axes Ox' and Oy it is possible to find such an azimuth that the axis of rotation ***e*** are the same, as for the reference axis Oz', which entails the same GZ-curves and the same heel angles $\eta = \phi' = \alpha'$. Equal heel angles (angles of rotation) are possible, when the vertical frame is perpendicular to the edge of intersection between the initial waterplane and the sea level, and when the wind impact screen passes through the said edge.

In ABS publications (Breuer & Sjölund, 2006 & 2009) the GZ-curve of minimum stability is found by the analysis of the dynamic arm l_d, as the function of the Euler's angles φ and Θ, related to the reference axis x'. For this purpose, iso-energy contours $l_d = const$ are used in the plane of the two said angles (Figure 16). Applying the method of the steepest descent path (SDP) it is possible to find a curve of the least dynamic arms, and thereby a curve of the least righting arms. They are both a function of the angle of rotation $\eta = \phi' = \alpha'$ of the rotation plane, unmentioned by the authors. The steepest descent method is complex, time consuming (it requires hundreds of calculation points for the ship longitudinally unbalanced), and entirely detached from the mechanism of inclinations with the least work. Nonetheless, it is identical with the GZ-curve of minimum stability for a freely floating object, as for the reference axis Oz'.

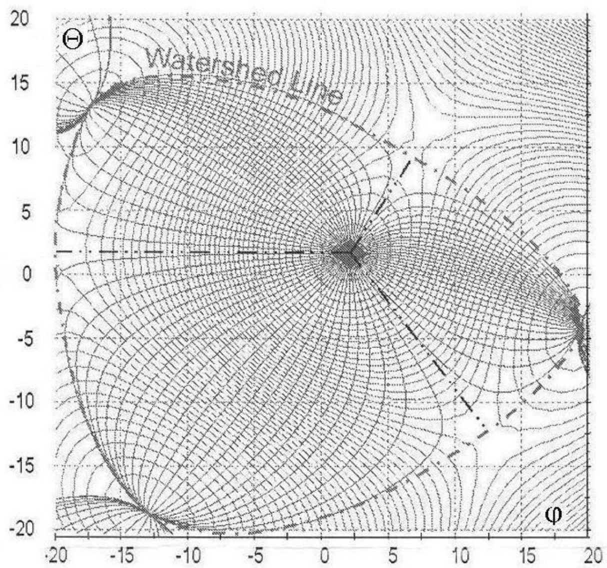

Figure 16. Steepest descent method (SDM)

Another possibility of calculating the GZ-curve with free trim is the *free twist* method applied by van Santen (2009 & 2013). In this method an axis of rotation $\boldsymbol{e} = \boldsymbol{w}$ is sought on the initial waterplane to be perpendicular to the righting arm l after rotation by a given heel angle α' around the axis. The GZ-curve thus obtained corresponds to the reference axis Oz'. Such a method, however, is not the most effective, particularly for large heel angles.

The GZ-curve of minimum stability can be best found as for a freely floating ship for the reference axis Oz', since the two curves are identical. For a given heel angle φ or α' the trim angle θ or ϑ' is found by iterations until the ship is longitudinally balanced, i.e., $\boldsymbol{e} \cdot \boldsymbol{r} = 0$, where the axis of rotation $\boldsymbol{e} = \boldsymbol{w}$ and the unit vector $\boldsymbol{n}$ are given by equations (14). The knowledge of the Euler's angles (the heel and trim angles) defines the unit vector $\boldsymbol{n}$, and this in turn defines the analytical angles φ and θ, essential for calculating the geometrical characteristics of the hull.

The curve of minimum stability can be obtained also with the help of the wind impact screen, described by the azimuth ψ. Two reference axes can be used: Ox'' and Oy'. For the first one, the rotation axis $\boldsymbol{e} = \boldsymbol{e}_2' \times \boldsymbol{n}$, where the unit vectors $\boldsymbol{e}_2'$ and $\boldsymbol{n}$ are given by equations (15), for the second, the rotation axis $\boldsymbol{e} = \boldsymbol{e}_1'$ and the unit vector $\boldsymbol{n}$ are given by equations (10) and (11), The latter quantity value defines the analytical heel angles φ and θ, essential for calculating the geometrical characteristics of the hull. The unit vector $\boldsymbol{n}$ depends on three degrees of freedom, dependent additionally on the azimuth ψ, whereas the axis of rotation $\boldsymbol{e}$ on two (in the case of the axis Ox'') or three (in the case of the axis Oy'). Hence, the condition of longitudinal balance $\boldsymbol{e} \cdot \boldsymbol{r} = 0$, for a given heel angle and azimuth defines the equilibrium trim. Knowing the three degrees of freedom the righting arm GZ and dynamic arm l_d can be obtained.

The determination of the GZ-curve with free trim is time consuming, since apart from balancing by iterations the displacement of the ship, we have to balance the ship longitudinally. The labour intensity can be drastically reduced by the Krilov–Dargnies method, which in a natural way tracks movements of the axis of floatation $\boldsymbol{f}$ during inclinations. In this method the new position of the ship is found without any iteration, making use of the differential properties of equi-volume waterplanes. The method implies that there is no room for the orthogonal tipping, understood as the loss of longitudinal stability during heeling the ship.

If we assume that in order to find the proper volume displacement and trim we need on average $4 \div 5$ iterations, then to find one point of the GZ-curve with free trim we need on average $4^2 \div 5^2 = 16 \div 25$ iterations. Hence, the Krilov–Dargnies method would be $16 \div 25$ times faster than buoyancy methods, which is worth considering.

6. NUMERICAL EXAMPLE

Based on the theory of a freely floating ship, presented here, the computer software WinSEA used in PRS for stability calculations has been modified by Dr. Andrzej Laskowski, the author of the program. The user can choose three modes of calculating the GZ-curve: 1) "engineering", related to the axis Ox' or Ox'', 2) "physical", related to the axis Oy or Oy', and 3) "natural", related to the z'-axis, identical with the curve of minimum stability. There is also a zero option of "maximum stability", for a ship with constant trim, normally not used.

Calculations for conventional ships show that the choice of the reference axis is meaningless. This is because for trims that occur the angle β between the traces of water in the PS and midships is virtually equal to the right angle. It yields the same rotation axes, independent of the reference axis. Hence, at the initial range up to the deck edge immersion, all the modes of calculations are virtually identical. The reason are small angles γ', even for the extremely asymmetric waterplanes. For example, for a rectangular waterplane with the ratio $L/B = 6$, which lost ¼ of the area, the angle γ' = 3.55°, although asymmetry of the waterplane is maximum (Pawlowski, 2013). This explains why the GZ-curve of minimum values at the initial range of stability cannot differ significantly from the remaining modes of calculations.

6.1 FISHING BOAT

For illustration, GZ-curves were calculated for a fishing boat and for a jack-up rig. More numerical examples can be found in the PRS report (Pawlowski, 2013). Main particulars of the fishing boat are as follows:

length between perpendiculars .... $L_{pp} = 23.9$ m,
breadth.. $B = 6$ m,
depth... $H = 3.1$ m,
design draught .... $T = 2.7$ m,
block coefficient .... $c_B = 0.63$.

The fishing boat has a transom stern of a long overhang and a large forecastle. Her GZ-curves are shown in Figure 17. Calculations were performed for a freely floating intact vessel in a partial loading condition, trimmed by stern, by the three modes of calculations, defined by the reference axes x', y, and z' in function of the appropriate angle of rotation η. In addition, calculations were performed for the ship with fixed trim, as in the position of equilibrium (curve c), and at level keel.

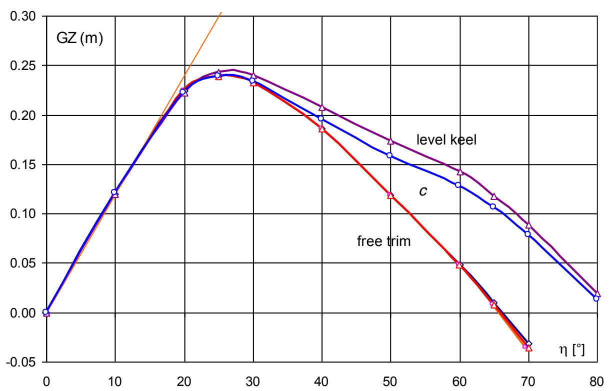

Figure 17. GZ-curves of the fishing boat

At the initial range of stability (up to the angle η_{max}, at which the GZ-curve reaches maximum) all the calculation modes yield the same results. The differences start above η_{max}. As expected, the highest values of the GZ-curves for large heel angles (in the sloping part) are obtained for the ship at level keel, greatly overestimating the range of stability. Somewhat smaller values are obtained for the ship with fixed trim, the same as at the initial position (curve *c*). As expected, both curves converge at the heel angle 90°. The least *GZ*-curves are obtained for the ship with free trim, wherein these curves are practically unaffected by the way the ship is balanced. They practically collapse into one curve.

6.2 JACK UP RIG

The situation is different for platforms at large heel angles, above the deck edge immersion. To see the effect of various reference axes on the *GZ*-curves, calculations were carried out for a jack-up rig. Its main particulars are these:

length .... $L = 58{,}1$ m,
maximum breadth .... $B_0 = 72{,}2$ m,
minimum breadth .... $B_1 = 14$ m,
depth .... $H = 7$ m,
draught .... $T = 4.65$ m,
waterplane coefficient of fineness .... $c_W = 0.597$,
height of centre of gravity above BP .. $KG = 24.37$ m.

This is a fictitious jack-up of simple geometric shape (Figure 18), conceived by ABS for testing calculations and widely investigated in literature (Santen van, 2009; Breuer & Sjölund, 2006 & 2009). *GZ*-curves are shown in Figure 19, while the run of trims in Figure 20–21. Calculations were performed for a damaged rig, trimmed by aft ($t = -2.058$ m), inclined to starboard with a heel of 1.73°.

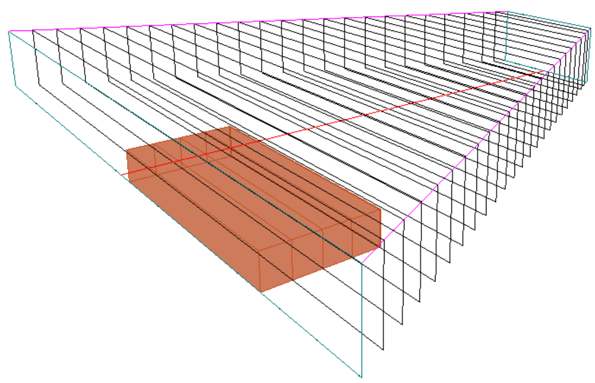

Figure 18. Generic platform

As can be seen from Figure 19, all the calculation modes yield practically the same *GZ*-curves at the initial range of stability. Above this range, the largest values correspond to the platform at level keel. The way of balancing has only a modest effect on the *GZ*-curves, and this can be taken as a rule. As discussed earlier, the *GZ*-curves for the reference axes *x* and *y* are virtually the same. Their range of stability is somewhat smaller than for the reference axis *z*'. Since the area under the curves has to be the same, the curve of larger range intersects with the curves of smaller range, and it has a smaller GZ_{max} value.

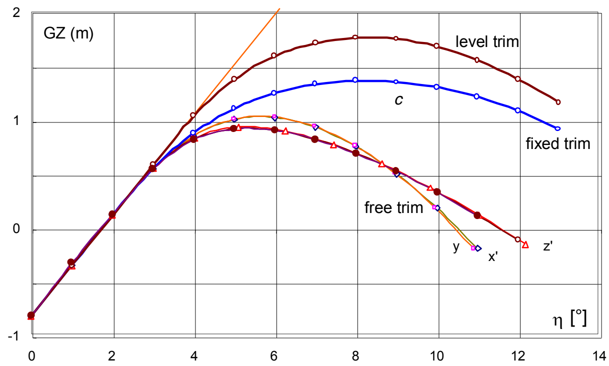

Figure 19. *GZ*-curves for the rig

Figure 20 shows the run of twist (trim) angle around the axis *z*' in function of the angle of heel α'. Because of an asymmetrical flooding and a small ratio *L*/*B*, these angles assume values larger by one order than for ships. But the waterplane is symmetric, therefore twist (rotation) of the platform starts above the angle at which the deck enters the water. For inclinations to portside the graph has a different character (Figure 21). The range of change of the twist angle for inclinations to portside equals 16°, while to starboard equals 26°. For heel angles α' < 7,4° to portside the twist angle is indefinite. It means that in the range α' ∈ ⟨−7,4°, 0⟩ the *GZ*-curve is indefinite.

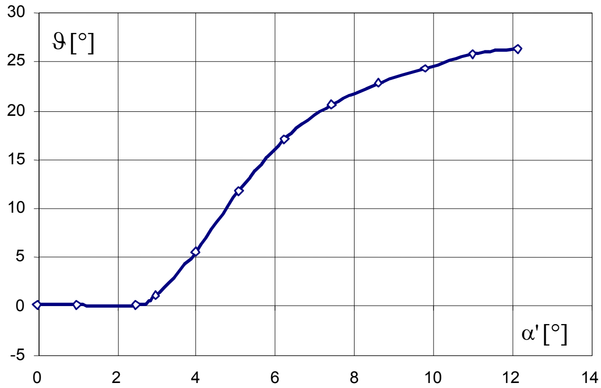

Figure 20. Run of twist around axis *Oz*' during heeling rig to starboard

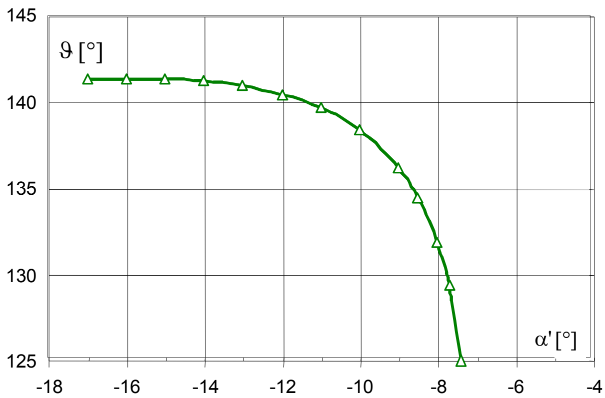

Figure 21. Run of twist around axis *Oz*' during heeling rig to portside

In the case of semisubmersible platforms, in view of small values of the ratio $L/B < 2$, the regulations require that the stability of platforms is analysed for various orientations relative to the wind direction, i.e. at various orientations of the rotation axis ***e*** relative to the principal axis of inertia of the initial waterplane, varying from 0° to 360°. It is not so much because of the *GZ*-curve but because of the wind heeling moment, strongly dependent on platform orientation relative to the wind (the windage area dramatically changes in the course of heeling). Calculating the wind heeling moment is not a problem, except for its cost. There are, however, problems with interpretation of the *GZ*-curve with free trim.

Equations (12) and (13) imply that the transverse and longitudinal components of the righting arm: l and l_e are functions of the angle of heel α' and twist $\Psi = \psi + \vartheta'$. The twist $\Psi = \Psi(\alpha')$ is a function of the angle of heel, resulting from the longitudinal equilibrium, i.e. from the solution of the equation $l_e(\alpha', \Psi) = 0$.

For a given heel angle $\alpha' = const$ there can be only a discrete number of twist angles Ψ at which a platform is in longitudinal balance. These angles can be easily found with the help of a graph $l_e = l_e(\alpha', \Psi)$ for a given heel angle α', as in Figure 22. As can be seen, there are four twist angles Ψ, corresponding alternately to minimum and maximum stability. The first angle corresponds to the absolute minimum of stability, while the last one – to the absolute maximum, which can be taken as a rule. These four equilibrium angles indicate that for a freely floating object only two meaningful orientations of the rotation axis ***e*** are possible, i.e. when it is parallel in an upright position to one or the other principal axis of inertia of the initial waterplane. The first orientation is the worst (some say the weakest); i.e. it yields the *GZ*-curve of the lowest arms. When the waterplane is asymmetric, the ship has to be inclined towards the initial heel. In the second orientation there are unstable inclinations of maximum potential energy.

Meanwhile, the regulations require the stability of platforms to be analysed at various orientations relative to the wind direction, described by the azimuth $\psi \in \langle 0°, 360° \rangle$, varying at every 5°. The azimuth is measured relative to the axis of rotation ***e***, perpendicular to the wind direction. Except for the four said orientations, i.e. ψ = 0°, 90°, 180° and 270°, in the remaining cases, if the ship is to be longitudinally balanced the *GZ*-curves for the reference axis *Oz*' are simply the same as for the azimuth ψ = 0, and for the other reference axes, the righting arms increase, assuming maximum values for the azimuth ψ = 90° and 270°. However, it is paid for by occurring of longer and longer intervals in which the ship cannot be longitudinally balanced (Figure 23). The said figure, identical for the reference axes *Ox*'' and *Oy*', illustrate at the same time the effect of azimuth on the righting arm $l \equiv GZ$ and dynamic arm l_d for a fixed value of the heel angle α' = 11°. It is worth noting that minima of the dynamic (righting) arm l_d have the same values and occur at the same azimuth, irrespective of the reference axis.

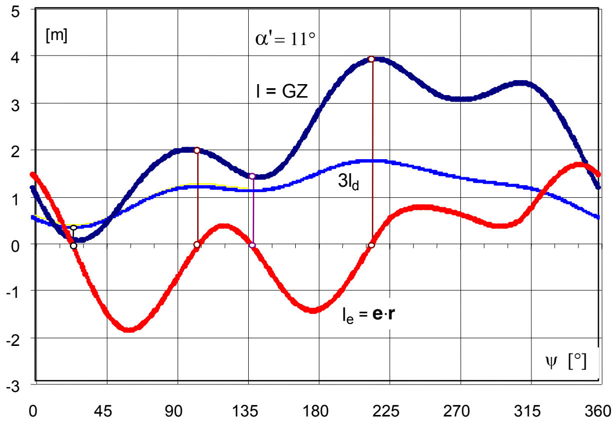

Figure 22. Run of stability characteristics l, l_e and l_d for rig versus azimuth Ψ for α' = 11° for reference axis *Oz*'

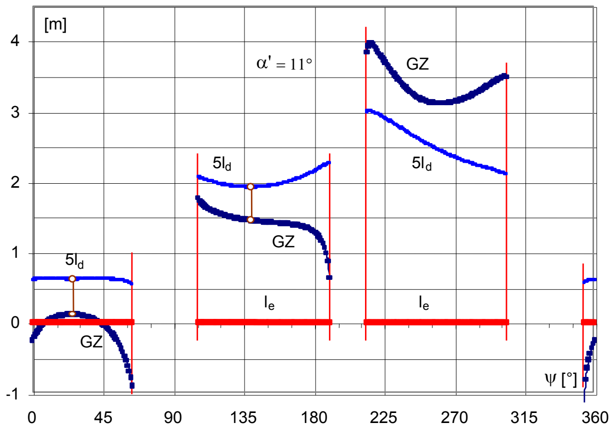

Figure 23. Run of the righting arm l and dynamic arm l_d for rig versus azimuth Ψ for α' = 11° for reference axis *Oy*'

Nonetheless, as prompted by regulations, the *GZ*-curves are calculated for any orientations. This is possible only, when the platform is longitudinally unbalanced or wrongly balanced. If the ratio *L*/*B* is too small, the lack of longitudinal balance can occur also for inclinations around the longitudinal axis, which makes it impossible to find the *GZ*-curve for all heel angles. The lack of balance for some heel angles does not mean that the platform rears, which is claimed in ABS publications (Breuer & Sjölund, 2006 & 2009). This phenomenon itself is termed there as *orthogonal tipping*. It is said that stability is then *fading*, in contrast to *vanishing stability*. The maximum trim, in the absolute values, depends on a given heel angle, normally does not exceed a dozen or so degrees. In our case for the heel angle α' = 6° the angle $\theta > -9{,}83°$, and for α' = 11° $\theta > -15{,}21°$. Orthogonal tipping does not occur in reality, which is self-explanatory in the light of the Krilov–Dargnies method.

It is noteworthy that the *GZ*-curve in Figure 22 has two additional extreme points, corresponding to points of inflexion on the curve of dynamic arms l_d. The reason for this strange behaviour is a small ratio of the principal moments of inertia of the waterplane in an upright position, equal merely 1.06. At the first equilibrium position the minima of the curves *GZ* and l_d are the lowest, which can be taken as a rule. Further, from Figure 22 it follows that a surface of dynamic arms $l_d = l_d(\alpha', \Psi)$, termed also as the energy to heel surface, should have two valleys (paths), corresponding to minimum l_d. Meanwhile, Figure 16 shows *three* paths (three minima). Admittedly, both figures correspond to different reference axes, but the choice of the reference axis has no significant effect on the dynamic arms.

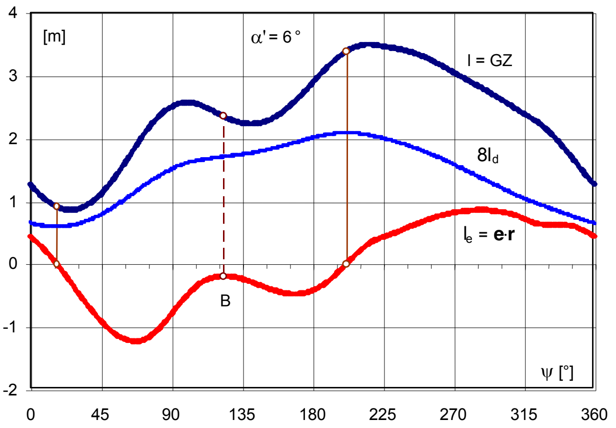

Figure 24. Run of stability characteristics l, l_e and l_d for rig versus azimuth Ψ for $\alpha' = 6°$

Figure 24 shows the run of stability characteristics for a different heel angle $\alpha' = 6°$. The run differs from that for the angle $\alpha' = 11°$ (Figure 22). The curve l_e has now only two zeros, instead of four. The zeros precisely coincide with the extremes of the curve of dynamic arms l_d but they are clearly shifted off from the extremes of the *GZ*-curve. Apex *B* of the curve l_e becomes tangent to the abscissa axis at the angle $\alpha' \approx 7{,}4°$. Hence, for heel angles $\alpha' > 7{,}4°$ there are again four zeros of the curve l_e (Figure 22), which is a condition for the existence of the *GZ*-curve for inclinations to portside. At the range $\alpha' < 7{,}4°$, i.e. $\alpha' \in \langle -7{,}4°, 0\rangle$ this curve does not exist (Figure 25), unless the rig turns by 180° around the axis *Oz*', assuming values as for heels to starboard. Due to a small ratio of the principal moments of inertia of the waterplane at an upright position and a larger asymmetry of flooding (a large negative righting arm at an upright position), the range in which the *GZ*-curve is indefinite due to lack of longitudinal balance, is exceptionally large, which results also from the approximated equation (35). For the reference axes *Ox*' and *Oy* the *GZ*-curves do not exist for heel angles below −13° (Figure 25–26). However, they are not the curves of minimum stability.

The angle of twist Ψ and *GZ*-curve for inclinations to portside are shown in Figure 21 and Figure 25. As discussed earlier, these characteristics exist for the angle $\alpha' < -7{,}4°$. They were obtained as readings for the third zero of the curve l_e, as in Figure 22. An identical curve can be obtained from direct calculations for inclinations to portside.

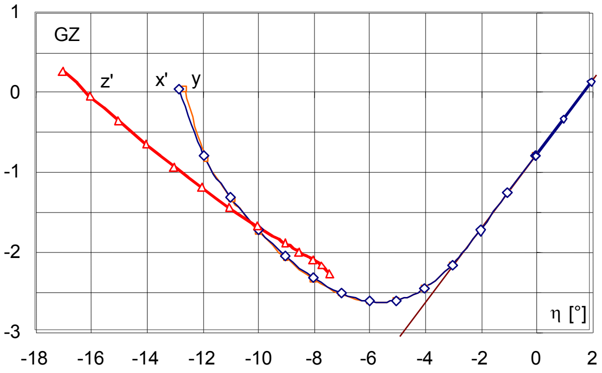

Figure 25. *GZ*-curves of rig for inclinations to portside

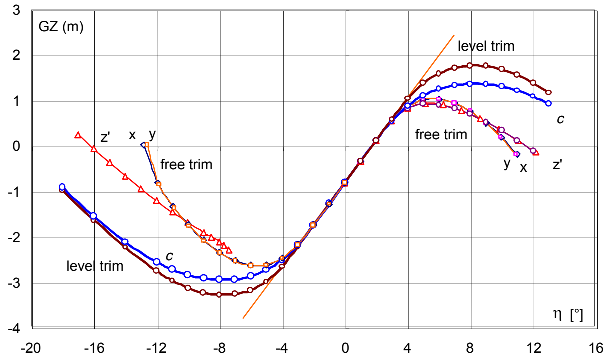

Figure 26. *GZ*-curves of rig for inclinations to both sides

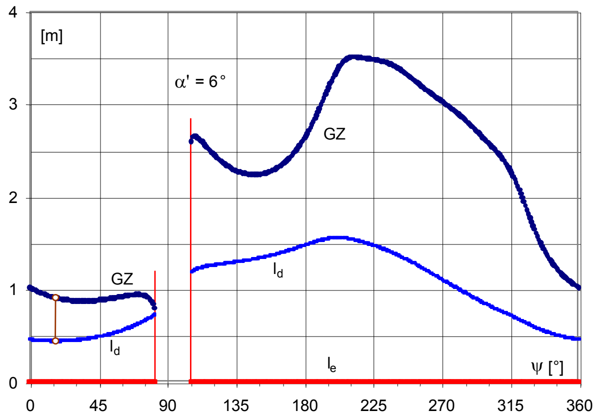

Figure 27. Run of righting arm *GZ* and dynamic arm l_d for rig versus twist Ψ for heel angle less than critical

Figure 27 shows the run of stability characteristics for a heel angle $\alpha' = 6°$ in function of the azimuth for the reference axis *Ox*''. We can see they are different from characteristics for the reference axis *Oz*' (Figure 24). Nonetheless, they both indicate the same features. Also here, due to the fact that the angle of heel $\alpha' = 6°$ is below the critical value 7.4°, a graph of the dynamic arm l_d in function of the azimuth has only one minimum. It defines a righting arm of the curve of minimum stability for the angle $\alpha' = 6°$ in the direction of the initial heel, identical with that in Figure 24. The lack of the second minimum means that for a heel on the other side the righting arm does not exist. As we know from the proceeding considerations, in the range of $\alpha' \in \langle -7{,}4°, 0° \rangle$ the *GZ*-curve does not exist, as the ship cannot be longitudinally balanced. From Figure 27 it follows additionally that for the azimuth at the range of $\psi \in \langle 86°, 106° \rangle$ the rig cannot be longitudinally balanced, if heel angle $\alpha' = 6°$.

In the case of asymmetrically flooded units the extremes of *GZ*-curves are somewhat shifted relative to the equilibrium position. It can be shown that there is no shift, if the principal axis of inertia of the waterplane is parallel to the axis of rotation $\boldsymbol{e}$. A proof is simple – we have to differentiate with respect to trim the righting arm $l \equiv GZ$, given by equation (17). Considering that the unit vector of the axis of rotation $\boldsymbol{e}$ need not be differentiated, we get the equation:

$$l' = \boldsymbol{e} \cdot (\boldsymbol{r}' \times \boldsymbol{n}) + \boldsymbol{e} \cdot (\boldsymbol{r} \times \boldsymbol{n}'),$$

where ' stands for the differentiation with respective to trim. It can be easily shown differentiating with respect to ϑ the unit vector $\boldsymbol{n}$, given by equation (17), that the vector $\boldsymbol{n}' = \sin\alpha\, \boldsymbol{e}$ is parallel to the axis of rotation $\boldsymbol{e}$, therefore the second term vanishes on the virtue of properties of the scalar triple product. Further, the vector $\boldsymbol{r}'$ has two components: longitudinal and transverse. A contribution to the triple product gives only the transverse component $\boldsymbol{r}'_T = -\boldsymbol{n} \times \boldsymbol{e} D''/V$, where the differentiation is with respect to trim τ, and D'' is the product of inertia of the waterplane in the $\xi''\eta''$ system (Figure 9), parallel to the axis of rotation $\boldsymbol{e}$. Hence,

$$\partial/_{\partial\tau}\, GZ = -D''/V.$$

It follows from the above equation that at the equilibrium position an extreme of the *GZ*-curve occurs, if $D'' = 0$, i.e. when the principal axis of inertia of the waterplane is parallel to the axis of rotation $\boldsymbol{e}$. In the case of ships, even damaged, the deviation of the principal axis of inertia from the axis of rotation $\boldsymbol{e}$ is small; therefore a shift of the extreme of *GZ* relative to the equilibrium position is imperceptible. In the case of damaged rigs with four zeros of the curve l_e, the shift is not large, but noticeable (Figure 22), whereas in the case of two zeros, a clear shift is visible (Figure 24).

7. CONCLUSIONS

Based on the results of theoretical and numerical analysis, the following conclusions can be drawn:

- a freely floating ship has minimum stability in the sense of the area under the *GZ*-curve. The said area is independent of the reference axes and is the smallest possible
- balancing of the ship does not change in space the direction of the righting moment, but decreases its value in proportion to the change of trim after balancing
- at the initial range of stability all the modes of calculations (including the mode of fixed trim) yield practically the same results
- for conventional ships the *GZ*-curves are independent of the reference axis (the way of balancing), while for platforms the effect is modest
- if the ship has an initial heel, the GZ-curve is indefinite in some one-sided neighborhood of zero, opposite to the initial heel, whose length increases with the initial heel. For ships, it is of the order of angular minutes, and for platforms – of the order of degrees. The azimuth (twist) of the unit in this range of heel is unstable, i.e., the unit can rotate automatically around the axis Oz' to assume a stable heel towards the initial heel
- for freely floating units only one *GZ*-curve is meaningful, related to the azimuth $\psi = 0$. For other azimuths, *GZ*-curves can have gaps in which they are indefinite
- the notion of cross-curves of stability is valid for a freely floating ship with minimum stability, when the ship's centre of gravity varies along the axis Oz', normal to the initial waterplane
- it is advisable to perform calculations of the *GZ*-curve by means of equi-volume waterplane method (Krilov–Dargnies), inclined around the instantaneous axis of floatation $\boldsymbol{f}$. It cuts radically the time of calculations (16÷25 times) in comparison to buoyancy methods, as it needs no iterations.

Hence, for ships there is no revolution – any method of calculating the *GZ*-curve with free trim yields virtually the same curve, identical with minimum stability. There is, however, a revolutionary conclusion for platforms – there is only one meaningful *GZ*-curve, related to transverse inclinations, as for the reference axis Oz'. In other words, for rigs there are no *GZ*-curves for various azimuths, required by regulations. In the case of the reference axis Oz' they are the same, irrespective of the azimuth, while for other reference axes they have, admittedly larger values but at the cost of unstable intervals, in which the ship cannot be longitudinally balanced. Hence, what sort of curves has been calculated? Either for rigs with fixed trim, or improperly balanced. The latter is very probable; as such notions as the reference axis, axis of rotation, plane of rotation, and angle of rotation of the plane of rotation are not mentioned in literature. Interesting papers, for instance (Santen van, 2009 & 2013; Breuer & Sjölund, 2006 & 2009) do not clearly state in which plane the rig was balanced.

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